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TECHNICAL REPORT

A METHOD FOR COMPUTING THE
INCOMPLETE BETA FUNCTION RATIO

by

A. R. DiDonato and M. P. Jarnagin, Jr.
Computation and Analysis Laboratory

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ABSTRACT

An efficient method is given for computing the incomplete beta function ratio, $I_x(a,b)$, on a high speed digital computer. The arguments a,b , are limited to positive integral multiples of one-half values over the ranges $1/2 \leq a \leq 10^8$, $1/2 \leq b \leq 60$.

The program has been coded in STRAP for the IBM 7030 (STRETCH). The average computing time for a ten decimal digit value of $I_x(a,b)$ is 2.6 milliseconds; on an IBM 7090 it would be about 8 milliseconds per case.

FOREWORD

The work reported in this publication was done in the Applied Mathematics Section of the Mathematics Research Group with Foundational Research funds No. R360FR103/2101/R0110101.

The development of an $I_x(a,b)$ routine was requested by Dr. K. Abt of the Mathematical Statistics Branch, Operations Research Division. The routine is of vital importance as a subroutine in a larger statistical program (NOVACOM), presently under development. NOVACOM, which will perform analysis of variance for data classifications with missing observations, is a program of wide applicability in weapons effectiveness studies and other statistical problems.

The IBM 7030 code was developed by Mr. Travis Herring from flow charts contained herein. Auxiliary subroutines for the calculation of certain elementary functions were taken from the 7030 Systems Library subroutines.

A NORC code for $I_x(a,b)$ was initially constructed primarily for exploratory type calculations. The auxiliary subroutines incorporated in the NORC code were taken from the library of NORC subroutines developed by Dr. A. V. Hershey of the Science Research Group.

This report contains more recent developments and supersedes NWL Report 1949 of 28 February 1965.

Date of completion was October 1966.

APPROVED FOR RELEASE:

/s/ RALPH A. NIEMANN, Acting
Technical Director

I. INTRODUCTION

The incomplete beta function, $B_x(a,b)$ is defined as follows:

$$B_x(a,b) \equiv \int_0^x t^{a-1} (1-t)^{b-1} dt, \quad (1)$$

where

$$0 \leq x \leq 1, \quad a > 0, \quad b > 0.$$

For $x = 1$, $B_x(a,b)$ is known as the complete beta function. It can be expressed in terms of three complete gamma functions, [2; p.127],

$$B_1(a,b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}, \quad (2)$$

where the complete gamma function, with argument s , is defined by

$$\Gamma(s) \equiv \int_0^\infty e^{-t} t^{s-1} dt, \quad s > 0. \quad (3)$$

Throughout this paper the following constraints are imposed on a and b :

{1} They can only take positive values of integral multiples of one-half.

{2} They satisfy the inequalities:

$$1/2 \leq a \leq 10^8, \quad 1/2 \leq b \leq 60.$$

The second constraint may be relaxed on the upper bound of b , at a proportional increase in the amount of calculation required.

The purpose of this report is to describe an efficient method on a digital computer for the high accuracy computation of the ratio of (1) to (2) subject to the constraints {1} and {2} on

a and b. This ratio is known as the incomplete beta function ratio and is indicated by the symbolism $I_x(a,b)$, i.e.,

$$I_x(a,b) \equiv B_x(a,b)/B_1(a,b) . \quad (4)$$

By the substitution of $u = 1 - t$ in (1)

$$I_x(a,b) = 1 - I_{1-x}(b,a) . \quad (5)$$

In probability terms, $I_x(a,b)$ is called the beta distribution function, with mean μ and variance σ^2 given by

$$\mu = a/(a+b) , \quad \sigma^2 = ab/[(a+b+1)(a+b)^2] , \quad [2; p 244]. \quad (6)$$

The importance of this function is reflected in Karl Pearson's monumental work, Tables of the Incomplete Beta Function, [10], which required ten years to complete (1923-1932). The method he employed will be outlined in Section III. The primary importance of the beta distribution function, $I_x(a,b)$, stems from the fact that it is directly related or interpreted in terms of three basic continuous probability distribution functions, the chi-square distribution, the F (variance ratio) distribution, and the Student's t distribution. It is also related to the discrete cumulative binomial distribution.

It will be shown in the subsequent discussion that the constraint {1} above is not a very severe one, since the important related distributions, just mentioned, are covered by the values of a and b allowed under {1}.

The remainder of this section is, for the most part, taken from [1; p 940-948]. Let X_1, X_2, \dots, X_v be independent and identically distributed random variables each following a normal distribution with mean zero and unit variance. Then, as it is

known in statistics, $X^2 \equiv \sum_{i=1}^v X_i^2$ is said to follow a chi-square

distribution with v degrees of freedom; the probability of the event $X^2 \leq \chi^2$ is given by

$$P\{X^2 \leq \chi^2\} = P(X^2 | v) = [2^{v/2} \Gamma(v/2)]^{-1} \int_0^{\chi^2} e^{-t/2} t^{(v/2)-1} dt . \quad (7)$$

A proof of (7) is given in [2; p 233].

Now, if X_1^2 and X_2^2 are independent random variables which follow chi-square distributions with ν_1 and ν_2 degrees of freedom respectively, then $X_1^2/(X_1^2 + X_2^2)$ follows a beta distribution where $a = \nu_1/2$, $b = \nu_2/2$. Thus

$$P\{X_1^2/(X_1^2 + X_2^2) \leq x\} = I_x(a, b) . \quad (8)$$

A proof of (8) is given in [2; p 243].

If we consider the same random variables X_1^2 , X_2^2 , then the distribution of the ratio

$$F = \frac{X_1^2/\nu_1}{X_2^2/\nu_2} \quad (9)$$

is said to follow the variance ratio or F distribution with ν_1 and ν_2 degrees of freedom. The probability that $F \leq F_0$ is given by $P\{F \leq F_0\} = P(F_0 | \nu_1, \nu_2)$

$$= \frac{\nu_1^{\nu_1/2} \nu_2^{\nu_2/2}}{B_1(\nu_1/2, \nu_2/2)} \int_0^{F_0} F^{\left(\frac{\nu_1}{2} - 1\right)} (\nu_2 + \nu_1 F)^{-(\nu_2 + \nu_1)/2} dF ,$$

$$F_0 \geq 0 . \quad (10)$$

A proof of (10) is given in [2; p 241-3]. The substitution $z = \nu_2/(\nu_2 + \nu_1 F)$ is applied to (10). It follows directly from this variable of integration substitution that

$$P(F_0 | \nu_1, \nu_2) = 1 - I_x\left(\frac{\nu_2}{2}, \frac{\nu_1}{2}\right) = I_{1-x}\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right), \quad (11)$$

where

$$x = \frac{\nu_2}{\nu_2 + \nu_1 F_0} , \quad 1 - x = \frac{\nu_1 F_0}{\nu_2 + \nu_1 F_0} .$$

If X_1 is a random variable following a normal distribution with mean zero and unit variance, and X^2 is a random variable following an independent chi-square distribution with ν degrees of freedom, the distribution of the ratio

$$X_1/\sqrt{X^2/\nu} \equiv t \quad (12)$$

is called the Student's t -distribution with ν degrees of freedom. The probability that $|t|$ will be less than a fixed constant, t_0 , is given by

$$\begin{aligned} P(|t| \leq t_0) &= A(t_0 | \nu) \\ &= \left[\sqrt{\nu} B_1\left(\frac{1}{2}, \frac{\nu}{2}\right) \right]^{-1} \int_{-t_0}^{t_0} (1 + t^2/\nu)^{-\frac{\nu+1}{2}} dt \quad (13) \end{aligned}$$

A proof of (13) is given in [2; p 237-240]. In terms of the beta distribution

$$A(t_0 | \nu) = 1 - I_x\left(\frac{\nu}{2}, \frac{1}{2}\right), \quad x = \frac{\nu}{\nu + t_0^2} \quad (14)$$

The derivation is straightforward; apply the transformation $z = \nu/(\nu + t^2)$ to (13).

In case a and b are specified as positive integers, the beta function is related directly to the cumulative binomial distribution, $E(n, r, x)$, which is given by

$$E(n, r, x) \equiv \sum_{i=r}^n e(n, i, x), \quad (15)$$

where

$$e(n, i, x) \equiv \binom{n}{i} x^i (1 - x)^{n-i}; \quad [21], [22] \quad (16)$$

If x is the probability of success in one trial, the cumulative binomial distribution, $E(n,r,x)$, represents the probability that at least r successes will occur in n independent trials, and $e(n,r,x)$ the probability that there will occur exactly r successes in n independent trials. In terms of I_x , we have

$$E(n,r,x) = \frac{B_x(r, n-r+1)}{B_1(r, n-r+1)} \equiv I_x(r, n-r+1) . \quad (17)$$

The derivations of (15) and (16) are given in [2; p 193-194]; (17) is derived in [21; p XVII]. Applications of $E(n,r,x)$ are also given in [21, p XXXIV]; applications of the continuous distributions given above can be found in many references, e.g., throughout [5].

The four distributions that have just been related to the beta distribution require only positive values of a and b at the integral multiples of one-half. Moreover, even the non-central F and t distributions are included by these values of a and b , [1; p 947].

A number of published tables exist for I_x or its inverse with respect to x . Four of the most extensive ones are referenced here and the ranges of the variables are given.

A table already mentioned is K. Pearson's, [10]. It is the most comprehensive for $I_x(a,b)$. The ranges of a, b , and x are $a = \frac{1}{2}(\frac{1}{2})50$, $b = \frac{1}{2}(\frac{1}{2})50$ such that $a \geq b$, $x = 0(.01)1.00$. Values of the beta ratio are printed to seven decimal digits.

For integer values of a and b there are the Tables of the Cumulative Binomial Probability Distribution, [21], issued by Harvard University in 1955, and the Tables of the Binomial Probability Distribution, [22], published by the Bureau of Standards in 1950. The ranges of n, r, x in [21] are:

$r = 0(1)n$, $n = 1(1)50(2)100(10)200(20)500(50)1000$,

$x = 0.01(0.01)0.50$, $1/16$, $1/12$, $1/8$, $1/6$, $3/16$, $5/16$, $1/3$, $3/8$, $5/12$, $7/16$.

$I_x(r, n - r + 1)$ is given to five decimal digits, where $a = r$, $b = n - r + 1$. The ranges of n, r, x in [22] are:

$$x = 0.01(0.01)0.50, \quad n = 2(1)49, \quad r = 1(1)n,$$

and $I_x(r, n - r + 1)$ is given to seven decimal digits.

A table of percentage points of $I_x(a, b)$ has been computed by C. M. Thompson, [15]. In this case, the variable x is tabulated as a function of I_x, a, b . The ranges are:

$$I_x = 0.005, .01, .025, .05, 0.1, .25, .5; \quad 2a = 1(1)30, 40, 60, 120, \infty;$$

$$2b = 1(1)10, 12, 15, 20, 24, 30, 40, 60, 120.$$

The computed value of x is given to five decimal digits.

In [9] a nomogram is given which yields graphical results for $I_x(a, b)$ somewhat beyond the ranges of [10] and [15]. The values of a and b extend to 70 and 60 respectively.

II. DIFFICULTIES IN COMPUTING $I_x(a,b)$

The importance of $I_x(a,b)$ makes it extremely desirable to have a digital computer program which is designed for the efficient calculation of $I_x(a,b)$ to say eight or more decimal digits for any values of a and b subject to the constraints on page 1. To the authors' knowledge, no efficient program exists for such a calculation. A description of a program which is suitable is given in Sections IV and V.

The next section includes a discussion of some previously published formulas, algorithms and computing programs. In order to more easily set forth where some of these methods fail to be useful, the major numerical difficulties in computing I_x are stated:

{a} A straightforward binomial expansion of the integrand in (1) and a subsequent integration to obtain an alternating series in powers of x cannot be used for large values of a and b . The eventual subtraction of consecutive terms of nearly equal absolute value causes a loss in significant digits which is prohibitive.

{b} $I_x(a,b)$ is a function of three independent variables. It is unlikely that one procedure or algorithm will suffice, and so it will be necessary to devise a variety of techniques over the ranges of a , b , and x which are contemplated.

{c} The extensive range of a , $\frac{1}{2} \leq a \leq 10^8$, introduces scaling problems in most procedures because terms of the order of $\Gamma(a)$ occur.

{d} The use of recurrence relations imposes the requirement of computing starting values, in which case, one is confronted with the evaluation of quantities such as $I_x(a, 1/2)$ for large a . This computation is not straightforward.

{e} Closely connected to {d} is the fact that one must dodge any procedure which attempts to sum over a elements, since this could entail the addition of 10^8 elements. Such a process would destroy the efficiency of the program and very likely the accuracy as well.

III. VARIOUS METHODS FOR COMPUTING $I_x(a,b)$, $I_x(\frac{1}{2}, b)$.

A search for finding a suitable set of methods for computing $I_x(a,b)$ was initiated by carrying out an investigation of the literature on the subject. In this section some of the more pertinent papers, from our point of view, are discussed, and reasons for not using a particular program or analysis are pointed out.

It seems fitting to begin with the algorithms used by K. Pearson in computing his table, [10]. They are founded on the recurrence relations:

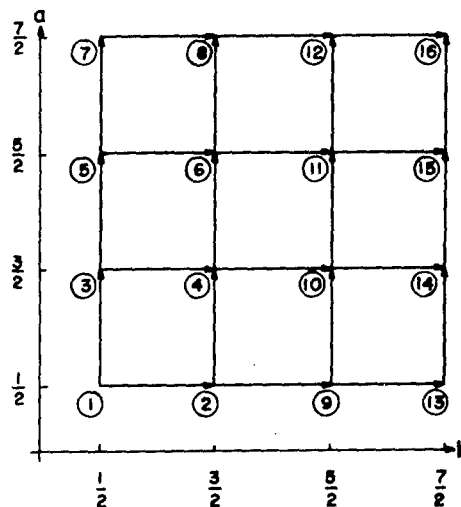
$$I_x(a,b) = x I_x(a-1,b) + (1-x) I_x(a,b-1), \quad (18)$$

$$I_x(a+1, 1/2) = I_x(a, 3/2) - \frac{2\Gamma(a+3/2)}{\sqrt{\pi} \Gamma(a+1)} x^a \sqrt{1-x}, \quad (19)$$

$$I_x(1/2, b+1) = I_x(3/2, b) + \frac{2\Gamma(b+3/2)}{\sqrt{\pi} \Gamma(b+1)} \sqrt{x} (1-x)^b, \quad (20)$$

which are derived in Appendix A.

The procedure is exemplified graphically. In Figure (1) the order of the computation for the case of integral multiples of one-half for a and b , where a and b extend as far as $7/2$, is shown. The ordered pair of values (a,b) at a node are those for which $I_x(a,b)$ is computed. The circled digits specify the order in which the consecutive values of I_x are obtained. Although this process is adequate for generating a table of I_x , it certainly would not be efficient if a exceeded 50 by any significant amount. Difficulties {d} and {e} of the last section would be encountered.



ORDER OF CALCULATION FOR
K. PEARSON'S TABLE

Figure 1

Numerous formulas have been developed and investigated by H. E. Soper and reported in [12]. His work includes perhaps a greater variety of relations for I_x than any other published paper on the subject. The contents include the rates of convergence of derived series for arbitrary real positive values of a and b . Polynomial fits and Fourier series expansions are considered. The general conclusion of Soper appears to be however, that none of the methods given, other than those used by K. Pearson are adequate for computation, [12; p 49].

A paper by J. Wishart, [20], resolves difficulty {c} for sufficiently large a and b . For completeness' sake, a derivation of his results is given.

By (1)

$$B_x\left(\frac{1}{2}, b\right) = \int_0^x (1-t)^{b-1} dt/\sqrt{t} = 2 \int_0^{\sin^{-1} \sqrt{x}} (\cos \theta)^{2b-1} d\theta, \quad (21)$$

where $t = \sin^2 \theta$. If $\sqrt{t} = 2y/(1+y^2)$, then

$$\begin{aligned} B_x\left(\frac{1}{2}, b\right) &= 4 \int_0^\lambda \left(\frac{1-y^2}{1+y^2}\right)^{2b-1} \left(\frac{dy}{1+y^2}\right) \\ &= 4 \int_0^\lambda \exp\left[-2N \sum_{i=1}^{\infty} (y^2)^{2i-1}/(2i-1)\right] \frac{dy}{1+y^2}, \end{aligned} \quad (22)$$

where

$$\lambda = \sqrt{x}/[1 + \sqrt{1-x}], \quad N = 2b - 1, \quad \left(\frac{1-y^2}{1+y^2}\right)^N = \exp\left[-N \ln\left(\frac{1-y^2}{1+y^2}\right)\right].$$

If $y = z/(2\sqrt{N})$ and if the power series in $z^2/4N$ is used for $\ln[(1 - y^2)/(1 + y^2)]$, then (22) becomes

$$B_x\left(\frac{1}{2}, b\right) = \frac{2}{\sqrt{N}} \int_0^{2\lambda\sqrt{N}} \frac{1}{1 + (z/2N)^2} \left\{ \exp \left[-2N \sum_{i=1}^{\infty} \left(\frac{z^2}{4N}\right)^{2i-1} / (2i-1) \right] \right\} dz \quad (23)$$

$$= \frac{2}{\sqrt{N}} \int_0^{2\lambda\sqrt{N}} e^{-z^2/2} \left(1 - \frac{z^2}{4N} + \frac{z^4}{16N^2} - \dots \right) \cdot \left(1 - \frac{z^6}{96N^2} - \frac{z^{10}}{2560N^4} + \frac{z^{12}}{18432N^6} + \dots \right) dz \quad (24)$$

$$\cong \frac{2}{\sqrt{N}} \int_0^{2\lambda\sqrt{N}} e^{-z^2/2} \left[1 - \frac{z^2}{4N} + \frac{z^4}{16N^2} - \frac{z^6}{32N^2} \left(\frac{1}{3} + \frac{1}{2N} \right) + \frac{z^8}{128N^3} \left(\frac{1}{3} + \frac{1}{2N} \right) - \frac{z^{10}}{960N^4} + \frac{z^{12}}{18432N^4} \right] dz \quad (25)$$

Equation (24) is obtained by expanding the product of exponentials in (23) in powers of z^2 . The approximation (25) is the result of truncating the power series in (24) to terms of order $1/N^4$. The indicated term by term integration of (25) and division by $B_1\left(\frac{1}{2}, b\right)$ gives

$$I_x\left(\frac{1}{2}, b\right) \cong \sqrt{2N} \frac{\Gamma(N/2)}{\Gamma((N+1)/2)} \left[m_0(\beta) - \frac{1}{4N} m_2(\beta) + \frac{3}{16N^2} m_4(\beta) - \frac{15}{32N^2} \left(\frac{1}{3} + \frac{1}{2N} \right) m_6(\beta) + \frac{105}{128N^3} \left(\frac{1}{3} + \frac{1}{2N} \right) m_8(\beta) - \frac{63}{64N^4} m_{10}(\beta) + \frac{1155}{2048N^4} m_{12}(\beta) \right], \quad (26)$$

where $\beta = 2 \sqrt{N} \lambda$ and the incomplete normal moment function,

$$m_1(\beta) \equiv \frac{1}{\sqrt{2\pi}} \int_0^\beta z^1 e^{-z^2/2} dz / (i-1)(i-3) \dots 2 \text{ or } 1. \quad (27)$$

Regrouping terms, Wishart's final result is obtained,

$$I_x\left(\frac{1}{2}, b\right) \cong \sqrt{2N} \frac{\Gamma(N/2)}{\Gamma([N+1]/2)} \left[\phi_0(\beta) - \frac{1}{N} \phi_1(\beta) + \frac{1}{N^2} \phi_2(\beta) - \frac{1}{N^3} \phi_3(\beta) + \frac{1}{N^4} \phi_4(\beta) \right], \quad (28)$$

where

$$\phi_0(\beta) \equiv m_0(\beta), \phi_1 \equiv (1/4) m_2(\beta), \phi_2(\beta) \equiv .1875 m_4(\beta) - .15625 m_6(\beta),$$

$$\phi_3(\beta) \equiv .234375 m_6(\beta) - .2734375 m_8(\beta),$$

$$\phi_4(\beta) \equiv .41015625 m_8(\beta) - .984375 m_{10}(\beta) + .56396484 m_{12}(\beta),$$

and

$$\beta \equiv 2 \sqrt{(2b-1)x} / (1 + \sqrt{1-x}). \quad (29)$$

Equation (28) has the very desirable feature of approaching the correct limiting value for $I_x\left(\frac{1}{2}, b\right)$ as $b \rightarrow \infty$. The equation

was not employed though, because it would have required incorporation into our program of an efficient normal probability integral subroutine. A fast subroutine for the probability integral generally requires storage of a set of function values at the expense of 300 to 500 storage locations in the computer. Also, a great deal more numerical analysis would have been required on (28) to fix rigorous error bounds and to determine the range over which it could be used efficiently. It is difficult to decide without the additional study whether it would be worthwhile to insert the procedure into our present program, especially since the procedure we employ for this calculation is quite efficient (see Sections IV and VI) in its own right.

H. E. Fettis, [3], treats the problem of evaluating

$\int_0^\theta \sin^N \phi \cos^M \phi d\phi$ numerically. This integral corresponds to

$\frac{1}{2} B_x\left(\frac{N+1}{2}, \frac{M+1}{2}\right)$ where $x = \sin^2 \theta$. It is emphasized at the

outset that it is believed Fettis was only interested in arbitrary small positive values of N and M . Nevertheless it was not obvious at first sight whether his formula would be useful for large N and M . As it turns out, they are not. The basic formula of the paper is given by

$$\begin{aligned} \int_0^\theta \sin^N \phi \cos^M \phi d\phi = & \frac{\sin^{N+1} \theta}{N+1} \left[1 - \frac{(M-1)(N+1)}{2(N+3)} \sin^2 \theta \right. \\ & \left. + \frac{(M-1)(M-3)(N+1)}{2^2 2!(N+5)} \sin^4 \theta - \dots \right] . \end{aligned} \quad (30)$$

If $\theta \sim \pi/2$ convergence is poor; in this case Fettis advocates interchanging N and M with an accompanying change in θ such that

$$\begin{aligned} \int_0^\theta \sin^N \phi \cos^M \phi d\phi &= \int_0^{\frac{\pi}{2}} \sin^M \phi \cos^N \phi d\phi \\ &\quad - \int_0^{\frac{\pi}{2}-\theta} \sin^M \phi \cos^N \phi d\phi . \end{aligned} \quad (31)$$

The first integral on the right in (31) is evaluated in terms of complete gamma functions, and the second integral is evaluated by (30) for $\phi = (\pi/2) - \theta$, which is small when $\theta \sim \pi/2$. Even so, if N and M are large and $\theta \sim \pi/4$, it does no good to carry out the interchange of N and M , and the convergence of (30) in this case would be slow. Difficulty {a} would be met, since consecutive

terms of nearly equal magnitude but opposite sign do occur. The associated loss in accuracy is easily seen for the example: $M = N = 99$, $\theta = \pi/4$. In this case, $I_{1/2}(50,50) = 1/2$, the integral on the left side of (30) is equal to 1.98×10^{-31} . The factor $\sin^{N+1} \theta / (N+1)$ in (30) is approximately 10^{-17} , and thus the second factor on the right hand side of (30) must be of order 10^{-14} . But, the first term in this second factor is unity, and the second term is negative and greater than unity in absolute value. Thus the second factor obviously approaches 10^{-14} necessarily through the addition of nearly equal consecutive terms with opposite sign.

For the special case of $M = 0$, ($b = 1/2$), Fettis sets $M = N$ in (30), and uses the fact that

$$\sin^N \theta \cos^N \theta = 2^{-N} \sin^N 2\theta$$

to derive

$$\int_0^{\bar{\theta}} \sin^N \phi \, d\phi = \frac{[2 \sin \bar{\theta}/2]^{N+1}}{N+1} \left[1 - \frac{(N-1)(N+1)}{2(N+3)} \sin^2 (\bar{\theta}/2) + \frac{(N-1)(N-3)(N+1)}{2^2 2! (N+5)} \sin^4 (\bar{\theta}/2) - \dots \right], \quad (32)$$

where $\bar{\theta} = 2\theta$. Equation (32) has the same deficiency for large N as the previous relation, (30). The values of $\bar{\theta}$ range from zero to $\pi/2$, and again nothing is gained in this case by using (31) for $\bar{\theta}$ near $\pi/2$. If $M = 0$ in (30), then

$$\int_0^{\bar{\theta}} \sin^N \phi \, d\phi = \sin^{N+1} \theta \sum_{i=0}^{\infty} \frac{(2i)!}{2^{2i} (i!)^2} \frac{\sin^{2i} \theta}{(N+2i+1)}. \quad (33)$$

The i^{th} term of this series is of order $(\theta^{-3/2})$ for $\theta \sim \pi/2$, and again (33) would not lead to an efficient algorithm for such θ , even though the terms of the series here are all positive.

M. E. Wise, [17], [18], [19], deals with the inverse problem, primarily, of finding good approximations to x , the percentage points of I_x , given I_x , a, b . Towards this objective, he

advances a formula for I_x in [17] which is derived by a contour integration in the complex plane. It is asymptotic for large a and b . Wise draws attention to papers by E. C. Molina, [8], and C. R. Rao, [11], and another of his own, [19], in which similar results are derived without resorting to the complex plane. Below we give our own derivation of Wise's result. The equation to be derived is given by (43).

Let

$$t = e^{-u/N}, \quad 0 \leq t < \infty, \quad 0 \leq x \leq 1, \quad (34)$$

where $N \equiv (a + b/2 - 1/2)$. Then substituting (34) into (1) and (2) and taking their difference gives

$$\begin{aligned} B_1(a,b) - B_x(a,b) &= \frac{1}{N} \int_0^y \exp \left[- \left(\frac{a-1}{N} \right) u \right] (1 - e^{-u/N})^{b-1} e^{-u/N} du \\ &= \frac{1}{N} \int_0^y e^{-u} (e^{u/2N} - e^{-u/2N})^{b-1} du \\ &= \frac{1}{N} 2^{b-1} \int_0^y e^{-u} [\sinh (u/2N)]^{b-1} du. \end{aligned} \quad (35)$$

The term $[\sinh (u/2N)]^{b-1}$ is expanded in powers of the argument $z = u/2N$. From [1; p 75, equation 4.3.71]

$$\ln \frac{\sinh z}{z} = \sum_{n=1}^{\infty} a_n z^{2n}, \quad |z| < \pi, \quad (36)$$

where $a_n \equiv \frac{2^{2n-1} B_{2n}}{n(2n)!}$, and B_{2n} is the $2n^{\text{th}}$ Bernoulli number, [1; p 810]. Hence

$$\sinh^{(b-1)} z = z^{b-1} \exp \left[(b-1) \sum_{n=1}^{\infty} a_n z^{2n} \right]. \quad (37)$$

It is tedious but not difficult to get the a_n 's for small values of n . Express each $\exp[(b-1) a_n z^{2n}]$ in its power series about $z = 0$ for $n = 1, 2, \dots, \ell$ and subsequently carry out the polynomial multiplications. The first six a_n are given by

$$\begin{aligned} a_1 = B_2 = 1/6, & \quad a_4 = -\frac{1}{2^3 \cdot 3^3 \cdot 5^2 \cdot 7} \\ a_2 = \frac{2^{4-1} B_4}{2 \cdot 4!} = -\frac{1}{180}, & \quad a_5 = \frac{1}{3^5 \cdot 5^2 \cdot 7 \cdot 11} \\ a_3 = \frac{2^5 B_6}{3 \cdot 6!} = \frac{1}{3^4 \cdot 5 \cdot 7}, & \quad a_6 = -\frac{1}{3^7 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13} \end{aligned} \quad (38)$$

Thus,

$$\exp[(b-1)a_n z^{2n}] = \sum_{i=0}^{\infty} \frac{[(b-1)a_n]^i}{i!} z^{2ni}; \quad n = 1, 2, \dots, \ell. \quad (39)$$

The product over n is taken to give

$$\begin{aligned} \exp\left[(b-1) \sum_{n=1}^{\infty} a_n z^{2n}\right] &= 1 + \frac{(b-1)}{2 \cdot 3} z^2 + \frac{(b-1)(5b-7)}{2^3 \cdot 3^2 \cdot 5} z^4 \\ &+ \frac{(b-1)}{3^3} \left[\frac{35b^2 - 112b + 93}{2^4 \cdot 3 \cdot 5 \cdot 7} \right] z^6 + \dots \end{aligned} \quad (40)$$

The substitution of (40) into (35) and the subsequent term by term integration of (35), with respect to u , gives

$$\begin{aligned}
 B_1(a,b) - B_x(a,b) \cong \frac{1}{N^b} & \left[\Gamma_y(b) + \frac{(b-1)}{24 \cdot N^2} \Gamma_y(b+2) \right. \\
 & + \frac{(b-1)(5b-7)}{2^7 \cdot 3^2 \cdot 5 \cdot N^4} \Gamma_y(b+4) \\
 & \left. + \frac{(b-1)(35b^2 - 112b + 93)}{2^{10} \cdot 3^4 \cdot 5 \cdot 7 \cdot N^6} \Gamma_y(b+6) + \dots \right], \quad (41)
 \end{aligned}$$

where

$$\Gamma_y(b) \equiv \int_0^y e^{-u} u^{b-1} du, \quad b > 0, \quad (42)$$

is known as the incomplete gamma function, [1; p 260]. The final result follows by dividing both sides of (42) by (2); thus for $\ell = 3$

$$\begin{aligned}
 I_x(a,b) \cong 1 - \frac{\Gamma(a+b)}{N^b \Gamma(a)} & \left[\frac{\Gamma_y(b)}{\Gamma(b)} + \frac{(b^3 - b)}{2^3 \cdot 3N^2} \frac{\Gamma_y(b+2)}{\Gamma(b+2)} \right. \\
 & + \frac{(b-1)(5b-7)}{2^7 \cdot 5 \cdot 3^2 N^4} \frac{\Gamma_y(b+4)}{\Gamma(b+4)} \\
 & \left. + \frac{(b-1)(35b^2 - 112b + 93)}{(2^{10} \cdot 3^4 \cdot 5 \cdot 7) N^6} \frac{\Gamma_y(b+6)}{\Gamma(b+6)} \right]. \quad (43)
 \end{aligned}$$

The ratio $\Gamma_y(b+2i)/\Gamma(b+2i)$ is computed by the recurrence relation

$$\begin{aligned}
 \Gamma_y(b+2)/\Gamma(b+2) &= [\Gamma_y(b)/\Gamma(b)] \\
 &= e^{-y} (1 + b + y) y^b / [b(b+1) \Gamma(b)]. \quad (44)
 \end{aligned}$$

This relation is derived by performing two integrations by parts on $\Gamma_y(b+2)$, as defined by (42), dividing the result by $\Gamma(b+2)$ and by using the gamma function equation

$$b \Gamma(b) = \Gamma(b+1) . \quad (45)$$

The series given by (43) is quite attractive from an asymptotic standpoint. Its rate of convergence from some numerical examples appears to be rapid for large a and b .

This series also was not incorporated into our program for a number of reasons. First, an efficient incomplete gamma function subroutine is needed. Such a subroutine does not seem to exist for the fast computing we require. Second, the storage requirements for such a routine might not be small. Finally, the computation time for (43) could be slowed down significantly if three or more terms are required, because of the cumbersome nature of the coefficients. Although it cannot be said with certainty, since the study of this phase was very limited, it appears that if (43) would be more efficient than the method we employ (see Sections IV and V), the difference would not be impressive as far as computing time. Certainly, (43) deserves further study.

In a paper by I. C. Tang, [14], a scheme is given for computing $B_x(a,b)$. The basic equation in his paper is similar to (48), (50) of this report. He has developed a series expansion with all positive terms in place of the usual alternating series for B_x . The derivation is elegant, however the relation itself was known to Soper, [12]. Two basic problems with which one is concerned, for large a and b , in the application of Tang's relations, i.e., difficulty {c}, the scaling problem, and difficulty {d}, the computation of starting values, are not discussed in [14].

This section is closed by a few comments on two digital computer programs published in the algorithm section of the Communications of the ACM.

W. Gautschi, [4], describes a program in which the scaling difficulty and the starting value problem are resolved. The basic relations are given by

$$\begin{aligned} I_x(a + n + 1, b) = & [1 + (n + a + b - 1) x / (n + a)] I_x(a + n, b) \\ & - [(n + a + b - 1) x / (n + a)] I_x(a + n - 1, b), \end{aligned} \quad (46)$$

$$\begin{aligned} I_x(a, b + n + 1) = & [1 + (n + a + b - 1) (1 - x) / (n + b)] I_x(a, b + n) \\ & - [(n + a + b - 1) (1 - x) / (n + b)] I_x(a, b + n - 1). \end{aligned} \quad (47)$$

Nevertheless, his program would not be suitable for our purpose when a and b are large, because of difficulty {e}. For example if $a > 60$ say 10^3 or 10^4 and b is approximately 60, it would require the computation and summing of 60 plus 10^3 or 10^4 terms of (46) or (47).

The other computer algorithm was designed by O. G. Ludwig, [7]. It was programmed at NWL for the IBM 7030 (STRETCH) by Mr. Robert Belsky in the interim period of development of the method described in Section IV. Ludwig's procedure worked quite well. In his procedure, four sums are generated in every case, whereas in the present method no more than four occur, and in some instances only one summation is required, e.g., when b is an integer. Moreover if $x > 1/2$ and a is large Ludwig's method requires summing over approximately the integer part of a elements. This leads to inefficiency for large a as mentioned previously.

The method for computing I_x , as described in the next section, was developed by the authors. Although it includes some relations in common with those mentioned in some of the preceding papers, it is basically a complete method in its own right, since it dispenses with all the difficulties given on page 7 satisfactorily, whereas none of the methods described in this section have this overall feature.

IV. AN EFFICIENT METHOD FOR COMPUTING $I_x(a,b)$

This section contains the main results of this report. The analysis that was developed for computing I_x is separated into 3 cases as follows:

- A - a or b is a positive integer no greater than 60;
- B - Neither a nor b is an integer, and $a \leq 60$;
- C - b is not an integer and $a > 60$.

The primary ideas or motives behind the method are:

(1) that a and b can be represented by k or $k-1/2$ and j or $j-1/2$, respectively where j and k are positive integers such that $1 \leq j \leq 60$, $1 \leq k \leq 10^8$;

(2) that all sums will be finite so no truncation error occurs, with two exceptions; in these cases the truncation error is rigorously and sharply bounded (See discussion on (81) and the evaluation of $\ln \Gamma(s)$ under Section V);

(3) that no procedure be used which requires summation over $k(a = k \text{ or } k - 1/2)$, unless $a \leq 60$;

(4) that no alternating power series are evaluated.

It will be assumed throughout that $I_x(a,b)$ is to be computed to an accuracy of $[[\log_{10} 1/\epsilon]]$ decimal digits, where ϵ is assigned and

$[[s]] \equiv$ greatest integer in s .

Case A: $b = j$, and/or $a = k \leq 60$ (See Flow Charts (1), (2))

If $b = j$, I_x can be computed from

$$I_x(a,b) = \sum_{i=1}^j a_i, \quad (48)$$

where

$$a_1 \equiv x^a \frac{\Gamma(a+i-1)}{\Gamma(a) \Gamma(i)} (1-x)^{i-1}, \quad (49a)$$

$$a_1 = x^a = I_x(a, 1). \quad (49b)$$

If $a = k \leq 60$, I_x may be computed by using (5), i.e.,

$$I_x(a, b) = 1 - \bar{I}_x(a, b) = 1 - \sum_{i=1}^k b_i, \quad (50)$$

where

$$b_i \equiv (1-x)^b \frac{\Gamma(b+i-1)}{\Gamma(b) \Gamma(i)} x^{i-1}, \quad (51)$$

and

$$\bar{I}_x(a, b) \equiv I_{1-x}(b, a). \quad (52)$$

The choice between (48) and (50) is made accordingly:

if $a \neq k$ (a not an integer), $b = j$, then use (48);

if $a = k \leq 60$, $b \neq j$ (b not an integer), then use (50);

if $a = k \leq 60$, $b = j$, then use (48) if $j \leq k$ and use (50) if $j > k$.

The derivation of (48) is given in Appendix A. Equations (50), (51) follow directly from (5) and (48).

The remainder of the analysis on Case A will be with respect to (48) since the results for (50) are analogous by the substitutions implied by (5).

A complication arises from (49a) because of the gamma functions. Although each a_i must remain less than $1 - \epsilon$ (otherwise, since all $a_i > 0$, $I_x \geq 1 - \epsilon$) the individual quantities $\Gamma(a+i-1)$ and $\Gamma(a)$, and even their ratio $\Gamma(a+i-1)/\Gamma(a)$, can exceed the value of the largest single precision number the computer can operate on. The same problem is manifest in the b_i and c_i coefficients given by (51) and (62), respectively. This difficulty with the a_i (and b_i) is resolved by the following scaling procedure:

Let

$$a_n \equiv \max_i a_i ,$$

then

$$\min_{n=1} \left\{ \left[\frac{(a-1)(1-x)}{x} \right] + 1, j \right\} \quad \begin{array}{l} k \neq 1 \\ k = 1 . \end{array} \quad (53)$$

The result given by (53) is easily deduced since, by the definition of a_n , it is required that

$$a_{n-1} \leq a_n , \quad 2 \leq n \leq j , \quad (54a)$$

$$a_{n+1} \leq a_n , \quad 1 \leq n \leq j-1 . \quad (54b)$$

Inequalities (54) imply

$$n \leq \left[\frac{(a-1)(1-x)}{x} \right] + 1 , \quad (55a)$$

$$n \geq \left[\frac{(a-1)(1-x)}{x} \right] , \quad (55b)$$

from which (53) follows. Inequalities (55) also imply that there are at most two a_n and if so they are consecutive.

Having found an expression for a_n , the $\ln a_n$ can be computed by

$$\begin{aligned} \ln a_n &= a \ln x + (n-1) \ln(1-x) \\ &\quad + \ln \Gamma(a+n-1) - \ln \Gamma(a) - \ln \Gamma(n) . \end{aligned} \quad (56)$$

Various sensings are made on $\ln a_n$ from which it may be usually concluded if $I_x \leq \epsilon$ or $I_x \geq 1 - \epsilon$. Thus all the a_i are under control, at this stage, since none is larger than $a_n = \exp [\ln a_n]$ which must remain less than $1 - \epsilon$ as explained above. The procedure is brought forth in detail in Flow Chart (2).

The $a_i (i \neq n)$ are computed by the following extremely simple and efficient recurrence relations:

$$a_{i+1} = \left(\frac{i + a - 1}{i} \right) (1 - x) a_i, \quad n \leq i \leq j - 1, \quad (57)$$

$$a_i = \left(\frac{i}{i + a - 1} \right) \left(\frac{1}{1 - x} \right) a_{i+1}, \quad 1 \leq i \leq n - 1, \quad (58)$$

which are easily derived from (49a).

The computation of (56) requires a method for evaluating $\ln \Gamma(s)$ directly, where s is used to represent the argument of any natural logarithm that appears in this report. The method by which this is accomplished is described in Section V, and by flow charts (6) and (7).

Case A is concluded by noting the following advantages:

- (1) All terms of the sums in (48) and (50) are of like sign.
- (2) The series to be summed are finite series with the number of terms to be summed not exceeding 60. Thus no truncation error need occur (actually one is introduced by a sensing in the program which permits truncation of the series if any of the a_i or b_i become less than specified tolerances. See Flow Chart (2).).
- (3) The magnitudes of the a_i are kept under control for any k such that $1/2 \leq a \leq 10^8$.
- (4) The procedure is efficient.

Case B: $a \leq 60$, $a = k - 1/2$, $b = j - 1/2$. See Flow Charts (1), (3), (4).

In this case $I_x(a,b)$ is computed from

$$I_x(a,b) = I_x(a,1/2) + \sqrt{1-x} \sum_{i=1}^{j-1} x^a \frac{\Gamma(a+i-1/2)}{\Gamma(a) \Gamma(i+1/2)} (1-x)^{i-1} \quad (59)$$

$$I_x(a,1/2) = I_x\left(\frac{1}{2}, \frac{1}{2}\right) - \sqrt{x} \sqrt{1-x} \sum_{i=1}^{k-1} \frac{\Gamma(i)}{\Gamma(i+1/2) \Gamma(1/2)} x^{i-1} \quad (60a)$$

$$I_x\left(\frac{1}{2}, b\right) = I_x\left(\frac{1}{2}, \frac{1}{2}\right) + \sqrt{x} \sqrt{1-x} \sum_{i=1}^{j-1} \frac{\Gamma(i)}{\Gamma(i+1/2) \Gamma(1/2)} (1-x)^{i-1}, \quad (60b)$$

$$I_x\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{2}{\pi} \tan^{-1} \left(\frac{\sqrt{x}}{\sqrt{1-x}} \right), \quad (61)$$

such that the arc tangent is between 0 and $\pi/2$. The derivations of (59), (60) are given in Appendix A.

We introduce the notation

$$c_i \equiv x^a \frac{\Gamma(a+i-1/2)}{\Gamma(a) \Gamma(i+1/2)} (1-x)^{i-1/2}, \quad (62)$$

$$d_i \equiv \frac{\Gamma(i)}{\Gamma(i+1/2) \Gamma(1/2)} x^{i-1}, \quad (63)$$

such that (59) becomes

$$I_x(a,b) = \frac{2}{\pi} \tan^{-1} \left(\frac{\sqrt{x}}{\sqrt{1-x}} \right) - \sqrt{x} \sqrt{1-x} \sum_{i=1}^{k-1} d_i + \sum_{i=1}^{j-1} c_i. \quad (64)$$

The terms d_i are generated by the simple recurrence relation

$$d_{i+1} = x \left[\frac{2i}{2i+1} \right] d_i, \quad 1 \leq i \leq k-2, \quad (65)$$

where

$$d_1 = \frac{2}{\pi}.$$

No scaling problem occurs with the d_i terms.

The c_i terms require scaling; it is done in the same way that the a_i were scaled.

Let

$$c_n \equiv \max_i c_i,$$

then

$$n = \min \left\{ \left[\left[(a-1)(1-x)/x + \frac{1}{2} \right] \right], j-1 \right\}, \quad (66)$$

where the result is derived from

$$\frac{c_{n+1}}{c_n} = \frac{n+a-1/2}{n+1/2} (1-x) \leq 1, \quad (67)$$

$$\frac{c_n}{c_{n-1}} = \frac{n+a-3/2}{n-1/2} (1-x) \geq 1; \quad (68)$$

the double square bracket notation was defined on page 19. It is known that c_n is less than $1 - I_x(a, 1/2)$ as otherwise $I_x(a, b)$ is equal to one. From this point, the scaling proceeds exactly as for the a_n . The recurrence relations for the c_i are:

$$c_{i+1} = \frac{i+a-1/2}{i+1/2} (1-x) c_i, \quad n \leq i \leq j-2, \quad (69)$$

$$c_i = \frac{i+1/2}{i+a-1/2} \left(\frac{1}{1-x} \right) c_{i+1}, \quad 1 \leq i \leq n-1. \quad (70)$$

The c_i are also summed in the same order that the a_i were, i.e., the c_i with $i > n$ are computed and summed in increasing order of i . Then the remaining c_i are computed and summed in decreasing order beginning at $i = n - 1$. Refinements in the program which are not included here may be gleaned from Flow Charts (3) and (4). The favorable factors listed for Case A on page 22 also apply for Case B, with the exception that two rather than only one sum, with as many as sixty terms, may have to be evaluated.

Case C: $a > 60$, $b = j - 1/2$. See Flow Charts (1), (3), (5).

Case C is by far the most difficult of the three cases to evaluate $I_x(a, b)$. The beta ratio is again given by (59), however (60) cannot be used for the computation of $I_x(a, 1/2)$ because the summation in (60a) runs over k , where this integer can be much larger than 60. Thus the problem here reduces to finding an efficient procedure for evaluating $I_x(a, 1/2)$ when a is large. After considering some of the methods proposed in the literature, [3], [12], [18], [20], it was decided to proceed by an entirely different approach, that of using Gaussian quadrature, [6; p 319]. This technique was chosen because the truncation error $E (\leq \epsilon')$ could be sharply and rigorously bounded, and moreover the error bound indicated that a surprisingly low order Gaussian formula would suffice for the accuracy desired. The details of the critical steps in the proofs required for the bound E' of the Gaussian error term are relegated to Appendix B, otherwise the analysis needed follows. We begin with some preliminaries.

Apply the transformation $t = 1 - u^2$ to $B_x(a, 1/2)$, so that

$$B_x(a, 1/2) = 2 \int_{\sqrt{1-x}}^1 (1 - u^2)^{a-1} du . \quad (71)$$

The following notation is introduced:

$$M \equiv B_1(a, 1/2) = \frac{\Gamma(a) \Gamma(1/2)}{\Gamma(a + 1/2)} - \sqrt{\frac{\pi}{a}} (a \rightarrow \infty) , \quad (72)$$

$$I_x(a, 1/2; \lambda) \equiv \frac{2}{M} \int_{\sqrt{1-x}}^{\lambda} (1 - u^2)^{a-1} du . \quad (73)$$

One can then write

$$I_x(a, 1/2) = I_x(a, 1/2; \lambda) + I_{1-\lambda}(a, 1/2; 1), \quad (74)$$

with the objective of making the last term in (74) small, i.e., for a given $\epsilon'' > 0$ to find a λ such that

$$I_{1-\lambda}(a, 1/2; 1) \leq \epsilon''. \quad (75)$$

A function $\lambda(\epsilon'')$ which satisfies (75) is given by (80). The derivation follows. From (73)

$$\begin{aligned} I_{1-\lambda}(a, 1/2; 1) &= \frac{2}{M} \int_{\lambda}^1 e^{(a-1) \ln(1-u^2)} du \\ &= \frac{2}{M} \int_{\lambda}^1 \exp \left[(a-1) \left(-u^2 - \sum_{i=2}^{\infty} \frac{u^{2i}}{i} \right) \right] du \end{aligned} \quad (76)$$

$$\begin{aligned} &\leq \frac{2}{M} \exp \left[(a-1) \left(-\sum_{i=2}^{\infty} \frac{\lambda^{2i}}{i} \right) \right] \int_{\lambda}^1 e^{-(a-1) u^2} du \\ &\leq \frac{2}{M} \exp \left[(a-1) \left(-\sum_{i=2}^{\infty} \frac{\lambda^{2i}}{i} \right) \right] \frac{1}{\sqrt{a-1}} \left[\int_{\lambda \sqrt{a-1}}^{\infty} e^{-z^2} dz \right. \\ &\quad \left. - \int_{\sqrt{a-1}}^{\infty} e^{-z^2} dz \right], \end{aligned} \quad (77)$$

where $z = \sqrt{a-1} u$. But the last integral in (77) is negligible for $a \geq 60$, since

$$\int_{\sqrt{a-1}}^{\infty} e^{-u^2} du \leq \frac{\sqrt{\pi}}{2} e^{-(a-1)}. \quad (\text{See [1], p. 298}).$$

Hence

$$\begin{aligned} I_{1-\lambda}^2(a, 1/2; 1) &\leq \frac{2}{M} \exp\left[(a-1) \left(-\sum_{i=1}^{\infty} \lambda^{2i}/i\right)\right] \frac{1}{\sqrt{a-1}} \int_{\lambda \sqrt{a-1}}^{\infty} e^{-z^2} dz \\ &\leq \frac{1}{M} \exp\left[(a-1) \left(-\sum_{i=1}^{\infty} \lambda^{2i}/i\right)\right] \frac{\sqrt{\pi}}{\sqrt{a-1}} e^{-\lambda^2(a-1)} \\ &= \frac{\sqrt{\pi}}{M \sqrt{a-1}} \exp\left[(a-1) \left(-\sum_{i=1}^{\infty} \lambda^{2i}/i\right)\right] \\ &= \frac{\sqrt{\pi}}{M \sqrt{a-1}} (1 - \lambda^2)^{a-1} \leq \epsilon'' \quad (78) \end{aligned}$$

By solving (78) for λ one obtains that

$$\lambda \geq \left\{ 1 - \left[\frac{M \sqrt{a-1}}{\sqrt{\pi}} \epsilon'' \right]^{\frac{1}{a-1}} \right\}^{1/2} \quad (79)$$

Inequality (79) is relatively sharp; it can be slightly improved provided one is willing to solve a transcendental equation for λ and accept the corresponding increase in computing time which would result.

The smallest value of λ is chosen from (79) for $\lambda(\epsilon'')$ so that

$$\lambda(\epsilon'') = \left[1 - \left(\frac{M \sqrt{a-1}}{\sqrt{\pi}} \epsilon'' \right)^{\frac{1}{a-1}} \right]^{1/2} \rightarrow \left[-\frac{1}{a-1} \ln \epsilon'' \right]^{1/2}, \quad (a \rightarrow \infty). \quad (80)$$

If a is large $\lambda(\epsilon'') = \lambda$ should be determined from its asymptotic form as given in (80). One can now deduce from (78) and (80) that the upper limit (unity) of the integral in (71) can be replaced by the smaller quantity $\lambda(\epsilon'') = \lambda$. If λ is less than $\sqrt{1-x}$, then the value of $I_x(a, 1/2)$ is less than ϵ'' ; a fact that is easily concluded from (73) and (74).

Having dispensed with these introductory results, the basic objective here of deriving a truncation error bound for the Gaussian integration procedure is now carried out.

The exact error term as a result of using Gaussian quadrature of order m , $O(m)$, [6; p 324], to numerically compute the integral of a function $f(t)$, with a sufficient number of derivatives, over $[-1, 1]$ is given by

$$E = \frac{2^{2m+1} (m!)^4}{(2m+1) [(2m)!]^3} f^{(2m)}(t_1), \quad -1 < t_1 < 1, \quad (81)$$

where $f^{(2m)}(t)$ means the $2m^{\text{th}}$ derivative of $f(t)$ with respect to t . The integral of (73) is transformed so that the limits of integration become -1 and 1 . The usual transformation

$$u = \frac{(\lambda - \sqrt{1-x})}{2} t + \frac{(\lambda + \sqrt{1-x})}{2} \quad (82)$$

applied to (73) gives

$$I_x(a, 1/2; \lambda) \cong \frac{\lambda - \sqrt{1-x}}{M} \int_{-1}^1 F(u) dt. \quad (83)$$

Similarly by applying the transformation

$$u = (\sqrt{1-x}/2) (1+t) \quad (84)$$

to $\bar{I}_x(a, 1/2)$, defined in (52), the result is

$$\bar{I}_x(a, 1/2) = \frac{\sqrt{1-x}}{M} \int_{-1}^1 F(u) dt, \quad (85)$$

where $F(u)$ represents the integrand in (73). It is important to consider \bar{I}_x here as well as I_x . The total interval of integration as specified in (73) is $(\lambda - \sqrt{1-x})$, however if we apply (83) only when $\lambda/2 < \sqrt{1-x}$ (and (85) only when $\lambda/2 \geq \sqrt{1-x}$), then essentially the total integration interval is never larger than $\lambda/2$ or half the maximum value of $(\lambda - \sqrt{1-x})$. This leads to a decrease in E' by a factor of $2^{+(2m+1)}$, since the integration interval appears in $f^{(2m)}(t)$ explicitly to the $(2m+1)$ power (see Equation 87). The term $\lambda/2$ is obviously never larger than $1/2$ and generally will be quite small. For example, if $a = 10^4$, $\epsilon'' = 9 \times 10^{-11}$, then $\lambda/2 = .024$ from (80).

The $2m^{\text{th}}$ derivative of f with respect to t is needed. The integrand from (83) and (85) is given by

$$f(t) = \begin{cases} \frac{(\lambda - \sqrt{1-x})}{M} F\left(\frac{\lambda - \sqrt{1-x}}{2}t + \frac{\lambda + \sqrt{1-x}}{2}\right), & \frac{\lambda}{2} < \sqrt{1-x}, \\ \frac{\sqrt{1-x}}{M} F\left(\frac{\sqrt{1-x}}{2}t + \frac{\sqrt{1-x}}{2}\right), & \frac{\lambda}{2} \geq \sqrt{1-x}. \end{cases} \quad (86a)$$

$$(86b)$$

Therefore, indicating the $2m^{\text{th}}$ derivative with respect to u of $F(u)$ by $F_u^{(2m)}$, $f^{(2m)}(t_1)$ in (81) is given by

$$f^{(2m)}(t_1) = \begin{cases} \frac{2}{M} \left(\frac{\lambda - \sqrt{1-x}}{2} \right)^{2m+1} F_u^{(2m)}(u_1), & \frac{\lambda}{2} < \sqrt{1-x}, \\ \frac{2}{M} \left(\frac{\sqrt{1-x}}{2} \right)^{2m+1} F_u^{(2m)}(u_1), & \frac{\lambda}{2} \geq \sqrt{1-x}, \end{cases} \quad (87a)$$

$$(87b)$$

where $u = u_1$ corresponds to $t = t_1$, and it is understood t_1 (and also u_1) is different in (87a) from (87b). The effect of reducing the integration interval is evident in (87). It is observed that the second factors on the right hand side of (87) are bounded by $(\lambda/4)^{2m+1}$. The principal result we wish to derive is the following expression for E' ,

$$E \leq E' \equiv \frac{2}{M} \left[\frac{(\lambda/2)^{2m+1} (m!)^3}{(2m+1) [(2m)!]^2} \right] \cdot \left[\frac{\Gamma(a)}{\Gamma(a-m)} \right] \leq \epsilon', \quad (88)$$

subject to the constraint that $a - 1 > 2m + 1/2$. Since $a > 60$ here and m will turn out to be in the neighborhood of ten, the constraint is insignificant for our application.

Let

$$U_{n,r} \equiv \frac{d^r}{du^r} [(1-u^2)^n], \quad 0 \leq u \leq 1, \quad (89)$$

so that

$$U_{a-1,2m} = F_u^{(2m)}. \quad (90)$$

It is shown in Appendix B that

$$U_{n,r} = \sum_{i=0}^{[[r/2]]} (-1)^{r-i} \frac{2^{r-2i} r! \Gamma(n+1)}{i! (r-2i)! \Gamma(n-r+i+1)} u^{r-2i} (1-u^2)^{n-r+i}, \quad (91)$$

and also that $U_{n,r}$ satisfies the following ordinary differential equation

$$(1 - u^2) U''_{n,r} + 2(n - r - 1)u U'_{n,r} + (r + 1)(2n - r) U_{n,r} = 0 \quad (92)$$

for positive integers r and real numbers $n \geq r$.

The key idea which leads to a useful bound on $U_{n,r}$ is that the absolute values of the extrema of $U_{n,r}$ form a decreasing finite sequence on $[0,1]$ for $n > r + 1/2$. Two closely related proofs of this statement are given in Appendix B. It is the crucial step in the sequence of steps employed for bounding E . Thus assuming the statement true, it follows

$$|U_{n,r}(u)| \leq |U_{n,r}(0)|, \quad r \text{ even.} \quad (93)$$

For $n = a - 1$, $r = 2m$, one obtains from (91)

$$|U_{a-1,2m}(0)| = \frac{(2m)! \Gamma(a)}{m! \Gamma(a - m)}, \quad (94)$$

and the desired result given by (88) follows.

The graphs on pages 36, 37 contain curves of $(-\log_{10} E')$ versus a based on (80) and (88), for given values of ϵ'' and m . Their purpose is to indicate the smallest order of Gaussian integration, $O(m)$, which can be used for a given ϵ' , where ϵ' represents the upper bound on E' . The results as graphically set forth clearly substantiate the remark made earlier that very low order Gaussian integration formulas will suffice for the evaluation of $I_x(a, 1/2)$ for large a . For example, in the computing program, as it is now operating $m = 10$ is used with $\epsilon'' = 9 \times 10^{-11}$, $\epsilon' = 4.5 \times 10^{-11}$, and it is apparent from the graphs that this value of m is adequate for all $a \in [60, 10^8]$.

The procedure by which the curves were constructed for a given ϵ'' was as follows:

- (α) a sequence of positive integers was chosen to represent various $O(m)$,
- (β) $\lambda(\epsilon'')$ was then computed by (80) for a sequence of values $a \in [60, 10^{40}]$, and a given value of ϵ'' ,
- (γ) these computed values of λ with their corresponding a values were then used in (88) to compute $\log_{10} E'$.

Thus the $O(m)$ to be used for a given ϵ' can usually be estimated conservatively from the graphs. A precise $O(m)$ can always be determined by computing a set of λ from (80) and the associated E' from (88) for various m and a . One observes that generally $\log_{10} E'$ is a very slowly increasing function of a .

This section is concluded with the explicit formulas used for the Gaussian quadrature of $I_x(a, 1/2)$. They are:

$$I_x(a, 1/2) \cong (\lambda - \sqrt{1-x}) \frac{\Gamma(a + 1/2)}{\Gamma(a) \Gamma(1/2)}$$

$$\cdot \sum_{i=1}^m w_i \left\{ 1 - \left[(\lambda - \sqrt{1-x}) \left(\frac{1+y_i}{2} \right) + \sqrt{1-x} \right]^2 \right\}^{a-1} + \frac{\epsilon''}{2} ,$$

$$\lambda^2 < 4(1-x) , \quad (95)$$

$$\bar{I}_x(a, 1/2) \cong \sqrt{1-x} \frac{\Gamma(a + 1/2)}{\Gamma(a) \Gamma(1/2)}$$

$$\cdot \sum_{i=1}^m w_i \left\{ 1 - \left[\sqrt{1-x} \left(\frac{1+y_i}{2} \right) \right]^2 \right\}^{a-1} ,$$

$$\lambda^2 \geq 4(1-x) , \quad (96)$$

where the y_i and w_i are the Gaussian abscissae and weights, respectively, of order m , $O(m)$, [1; p 916]. Since the last term in (74) is always non-negative and no larger than ϵ'' for λ which satisfies (80), it follows that

$$\left| I_x(a, 1/2) - [I_x(a, 1/2; \lambda) + \epsilon''/2] \right| \leq \epsilon''/2, \quad (97)$$

since

$$I_x(a, 1/2) \geq I_x(a, 1/2; \lambda). \quad (98)$$

This accounts for the additional $\epsilon''/2$ in (95).

The w_i and $(1 + y_i)/2$ are tabulated below for $O(10)$ to 14 significant digits on $[-1, 1]$, where y_i are the Gaussian abscissae and w_i the Gaussian weights.

$(1 + y_i)/2$	w_i
0.0130 4673 5791 414	0.0666 7134 4308 688
0.0674 6831 6655 507	0.1494 5134 9150 58
0.1602 9521 5850 49	0.2190 8636 2515 98
0.2833 0230 2935 38	0.2692 6671 9310 00
0.4255 6283 0509 18	0.2955 2422 4714 75
0.5744 3716 9490 81	0.2955 2422 4714 75
0.7166 9769 7064 62	0.2692 6671 9310 00
0.8397 0478 4149 51	0.2190 8636 2515 98
0.9325 3168 3344 49	0.1494 5134 9150 58
0.9869 5326 4208 59	0.0666 7134 4308 688

The Flow Chart (5) covers this part of the program.

The use of (59), with $I_x(a, 1/2)$ precomputed, as above, gives $I_x(a, b)$.

The quantities ϵ , ϵ' , and ϵ'' which appeared in this section are briefly summarized. The number ϵ is specified slightly less than $5 \times 10^{-(\rho+1)}$ where ρ is the number of decimal digits to which $I_x(a, b)$ is to be computed. The number ϵ' is specified and is used for bounding the truncation error due to Gaussian quadrature which is used to evaluate $I_x(a, 1/2)$ when $a \geq 60$. The graphs on pages 36-37 are a guide to determine the $O(m)$ for given ϵ' , ϵ'' . Generally ϵ' is taken equal to $\epsilon/4$. The number ϵ'' is taken equal to $2(\epsilon - \epsilon')/3$. This number is used in (59) to bound the c_n . The details are shown in Flow Chart (3). The quantity ϵ'' is also used in (80) to reduce the Gaussian interval of integration from $[\sqrt{1-x}, 1]$ to $[\sqrt{1-x}, \lambda]$.

The ϵ -quantities are used primarily to insure that $I_x(a, b)$ is computed within ϵ when (59) is used. Thus if $\epsilon' = \epsilon/4$ and $\epsilon'' = 2(\epsilon - \epsilon')/3 = \epsilon/2$, the following analysis shows that the required accuracy is attained.

$$I_x(a, b) = I_x(a, 1/2; \lambda) + I_{1-\lambda}^2(a, 1/2; 1) + \sqrt{1-x} S, \quad (99)$$

where S denotes the summation of $j-1$ terms in (59); and a similar relation holds between the computed quantities, distinguished by asterisks from the corresponding true values in (99). Then, taking differences and using the triangle inequality,

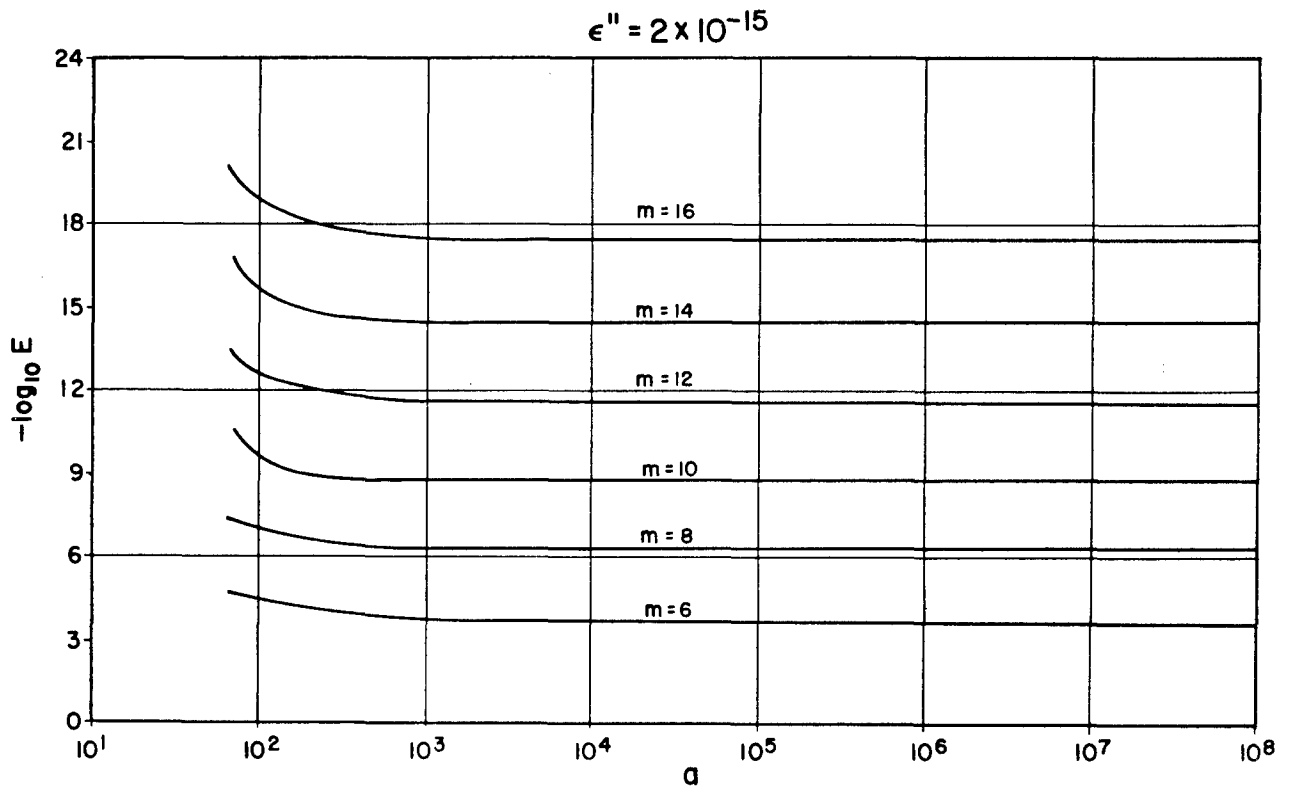
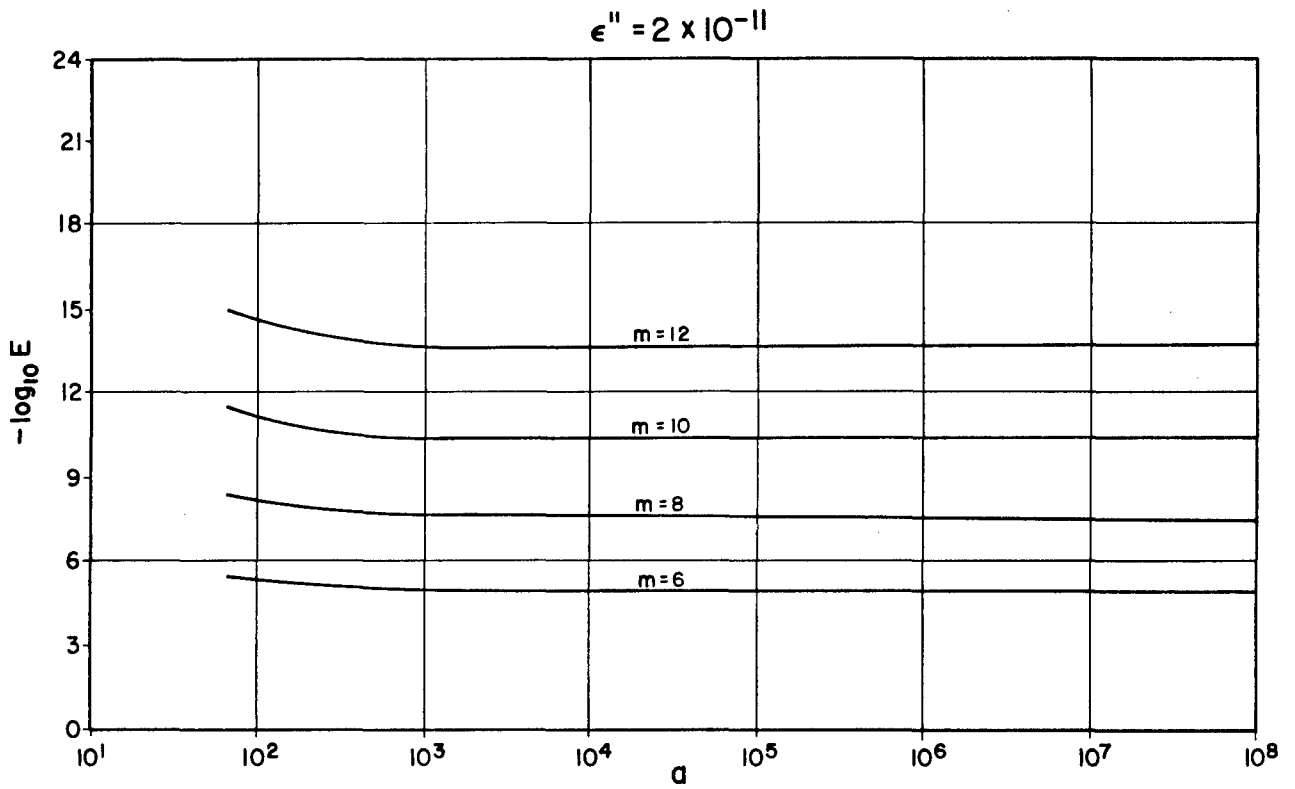
$$\begin{aligned} |I_x(a, b) - I_x^*(a, b)| &\leq |I_x(a, 1/2; \lambda) - I_x^*(a, 1/2; \lambda)| \\ &+ |I_{1-\lambda}^2(a, 1/2; 1) - I_{1-\lambda}^{*2}(a, 1/2; 1)| + \sqrt{1-x} |S - S^*|, \end{aligned} \quad (100)$$

where the value given to $I_{1-\lambda}^{*2}(a, 1/2; 1)$ is explained below.

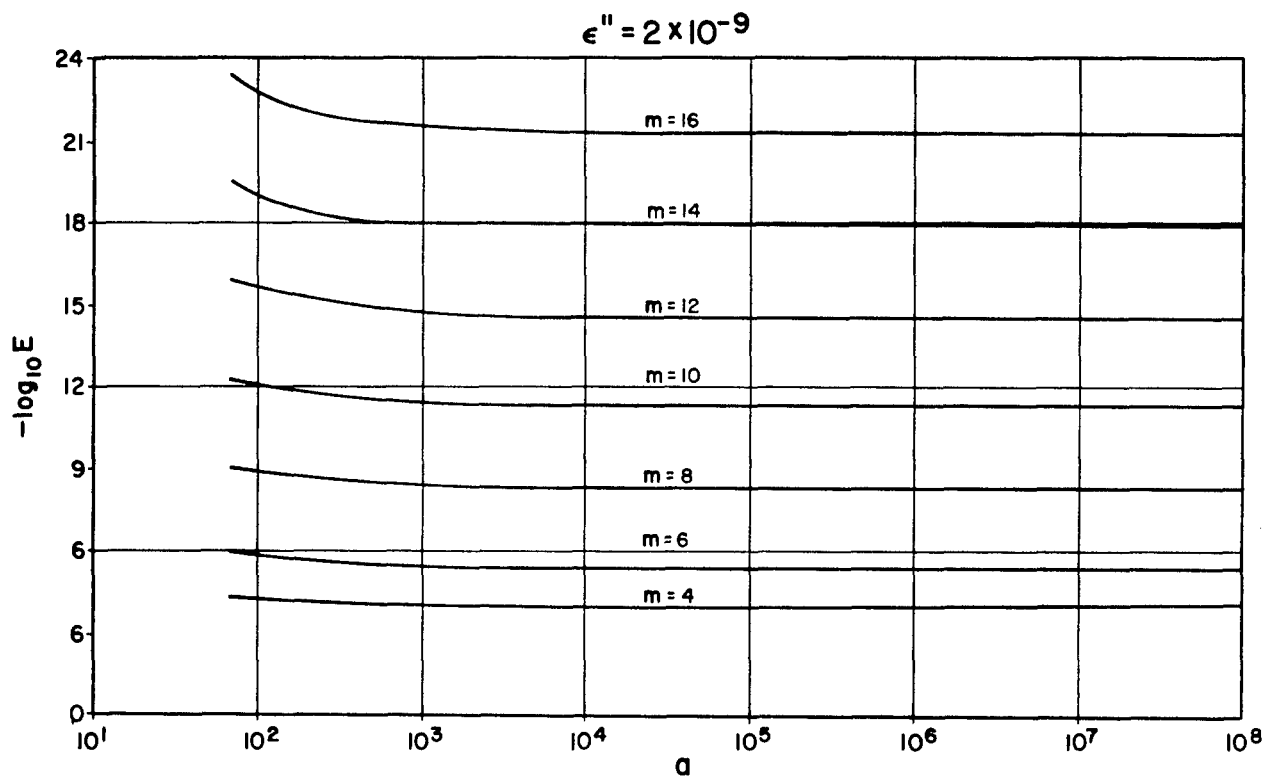
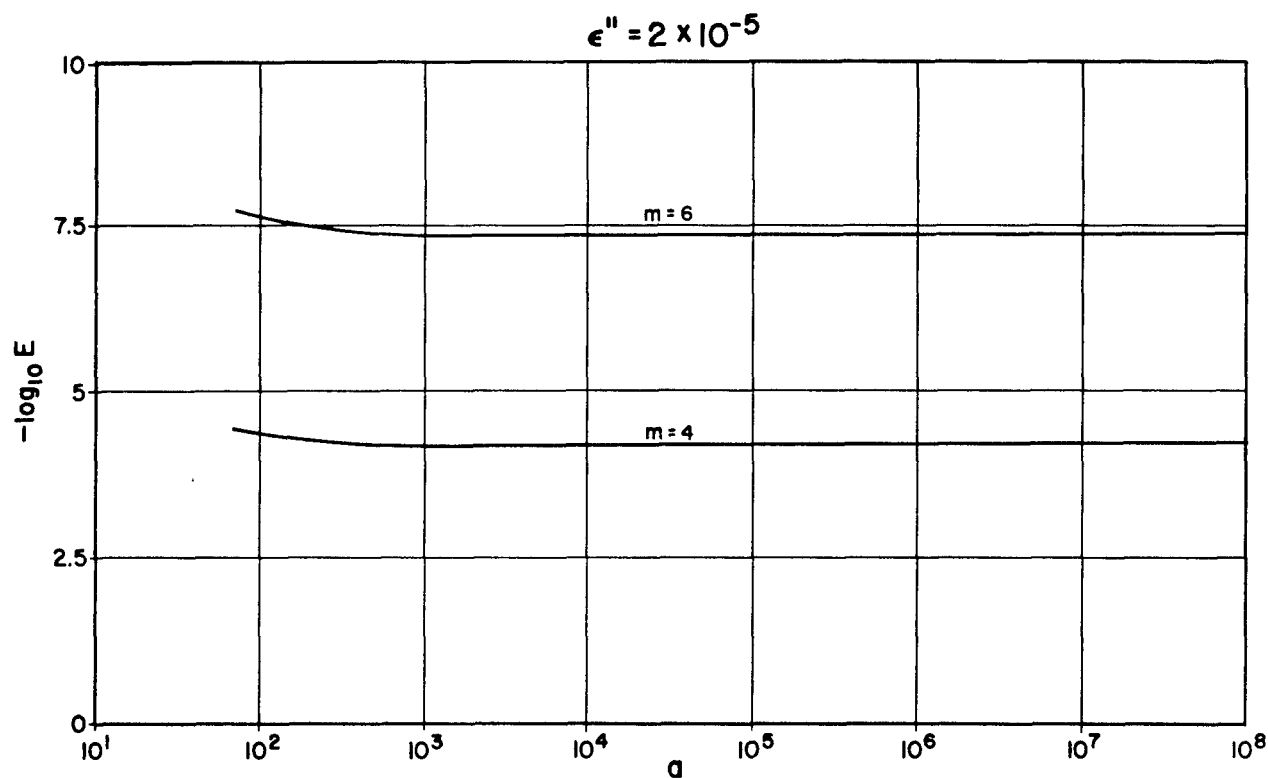
But the first term on the right in (100) does not exceed ϵ' , or $\epsilon/4$, through the choice of the proper order of Gaussian integration. Now $0 \leq I_{1-\lambda}^2(a, 1/2; 1) \leq \epsilon''$, by (78). Hence, reasoning as in (97) and (98), we arbitrarily take $I_{1-\lambda}^2(a, 1/2; 1)$ as $\epsilon''/2 = \epsilon/4$, and this guarantees that $|I_{1-\lambda}^2(a, 1/2; 1) - I_{1-\lambda}^{*2}(a, 1/2; 1)| \leq \epsilon/4$. The last term in (100) does not exceed ϵ'' or $\epsilon/2$, as shown by the method of determining the number of terms computed in the summation (see Flow Charts (3) and (4)). Thus $|I_x(a, b) - I_x^*(a, b)| \leq \epsilon/4 + \epsilon/4 + \epsilon/2 = \epsilon$, as was to be shown.

The program is presently set for obtaining $I_x(a, b)$ to within two units in the tenth decimal digit for $a \leq 10^8$. The ϵ -quantities are specified by

$$\epsilon = 1.8 \times 10^{-10}, \quad \epsilon' = 4.5 \times 10^{-11}, \quad \epsilon'' = 9.0 \times 10^{-11}.$$



Gaussian Quadrature Error Vs. a For $I_x(a, 1/2)$ At Fixed Values Of m



Gaussian Quadrature Error Vs. a For $I_x(a, 1/2)$ At Fixed Values Of m

V. COMPUTATION OF $\ln \Gamma(s)$, $K \equiv \ln \Gamma(a + c) - \ln \Gamma(a)$

The cases A, B, C which have been described above require the computation of $\ln \Gamma(s)$ or K to high accuracy, where s represents a positive integral multiple of one-half and c takes the values $n - 1$ or $1/2$. This is dealt with in an efficient manner, at the expense of two hundred storage locations, by storing the value of $\ln \Gamma(s)$ for $s = 1/2(1/2)100$ to the full accuracy of a single precision number (which is fourteen significant digits on STRETCH) and by using asymptotic series for $\ln \Gamma(s)$ or K when $s > 100$.

In such cases, it would seem convenient to always use the asymptotic series, for $\ln \Gamma(s)$, [1; p 257], which is given by

$$\begin{aligned} \ln \Gamma(s) \cong & (s - 1/2) \ln(s - 1) - (s - 1) + (1/2) \ln 2\pi \\ & + \frac{1}{12} \frac{1}{s - 1} - \frac{1}{360} \frac{1}{(s - 1)^3} + \frac{1}{1260} \frac{1}{(s - 1)^5} - \dots \quad (101) \end{aligned}$$

where the sum of the first five terms is sufficient for thirteen decimal digit accuracy with $s > 100$. It is observed however that in every case where $\ln \Gamma(s)$ is needed actually the difference K appears. The use of (101) to compute the two logarithmic terms of K separately leads to a prohibitive loss of significant digits if a is very large. This may be seen by observing that the dominant term in (101) for $s = a + c$ or a is of the order of $a \ln a$. Thus, upon subtraction, the undesirable loss of digits occurs, e.g., if $a = 10^4$ and $c = 1/2$ four digits are lost. If $a = 10^8$, $c = 1/2$ then $\ln \Gamma(10^8 + 1/2) - \ln \Gamma(10^8) = 1742068075.3142 - 1742068066.1038 = 9.2104$, so that in this case nine digits are lost.

This difficulty is resolved by introducing the following asymptotic series for K , if $a > 100$,

$$\begin{aligned} \ln \Gamma(a + c) - \ln \Gamma(a) \cong & c - \frac{1}{2} \frac{c}{a} \left[\frac{\ln(1 + c/a)}{c/a} - 1 \right] \\ & - \frac{1}{2} \frac{c}{a} + c \ln(a + c) - \frac{1}{12} \left[\frac{1}{a} - \frac{1}{a + c} \right] \\ & + \frac{1}{360} \left[\frac{1}{a^3} - \frac{1}{(a + c)^3} \right] - \frac{1}{1260} \left[\frac{1}{a^5} - \frac{1}{(a + c)^5} \right] + \dots \quad (102) \end{aligned}$$

The series can be derived by the use of the standard Stirling approximation to $\ln \Gamma(s)$. The first expression in square brackets on the right hand side of (102) is evaluated by the series

$$\frac{\ln(1 + \theta)}{\theta} - 1 = -y + \frac{2y^2}{2 + \theta} \left[\frac{1}{3} + \frac{y^2}{5} + \frac{y^4}{7} + \frac{y^6}{9} + \dots \right], \quad (103)$$

where $\theta \equiv c/a$, and $y \equiv \theta/(2 + \theta)$. The series in (103) is easily and efficiently generated by the recurrence relation

$$A_p = \left(\frac{2p + 1}{2p + 3} \right) y^2 A_{p-1}, \quad p = 1, 2, \dots, \quad (104)$$

where

$$A_p \equiv y^{2p} (2p + 3), \quad A_0 = 1/3. \quad (105)$$

Either five or ten terms of this series is used to attain fourteen digit accuracy such that if

1. $0 < \theta \leq 0.15$ five terms are used;
2. $0.15 < \theta < 0.6$ ten terms are used.

It is also necessary to retain the series given by (101) for those cases when the value of $\ln \Gamma(a + c)$ is not stored and yet $a < 100$, e.g., if $a + c = 140$ and $a = 90$. In such cases no significant loss of digits will occur in computing the two logarithmic terms of K separately.

Thus if $a + c \leq 100$, K is obtained by table look-up. If $a + c > 100$ but $a < 100$ then $\ln \Gamma(a + c)$ is computed from (101) and $\ln \Gamma(a)$ by table look-up. If $a > 100$ then K is computed from (102) and (103). The details are given in flow charts ⑥, ⑦.

VI. COMPUTING PROGRAM FOR $I_x(a,b)$ - FLOW CHARTS

The numerical calculation of $I_x(a,b)$ for

$$\begin{aligned} \frac{1}{2} \leq a \leq 10^8, & \qquad \frac{1}{2} \leq b \leq 60, \\ a = k, \text{ or } a = k - \frac{1}{2}, & \qquad b = j, \text{ or } b = j - \frac{1}{2}, \end{aligned}$$

where j and k are positive integers, is based on equations (48), (50), (64), (80), (95), (96), (97), (101), (102), (103) of the last two sections. The program, as outlined on the Flow Charts ① - ⑦, has been coded, as a subroutine, for the NORC and the IBM 7030 (STRETCH) in absolute machine language. Mr. Travis Herring prepared the STRETCH coding.

The inputs to the program are $a, b, x, \epsilon, \epsilon'$. The ϵ -quantities are discussed on pages 34-35 of the last section. If the number of decimal digits required in $I_x(a,b)$ is other than 2 units in the tenth decimal place, then this could necessitate a change in the number of Gaussian multipliers required as a result of a change in ϵ' and ϵ'' .

Two constants appear in the flow charts which depend on ϵ . They are identified by the letters f and g and are defined by

$$\begin{aligned} f &= \ln \epsilon'' = \ln 2(\epsilon - \epsilon')/3 \\ g &= \ln \epsilon \end{aligned} \tag{106}$$

Generally, the notation used in the flow charts allots lower case letters to numerical values and identifies the machine location in which that value is stored by the corresponding upper case letter; thus the storage location for the number σ would be Σ .

There are a total of seven flow charts starting with the master flow chart in which the over-all computing procedure is outlined.

The average computing time on STRETCH is 2.6×10^{-3} seconds per case. This would mean that an average computing time per case on an IBM 7090 would be about 8 milliseconds. The average time on STRETCH was determined by running large sets of cases for random choices of x values. Also very many cases were run by taking equal increments in the variables a, b, x and essentially spanning the space generated by these variables.

It is easily observed from Flow Charts ① and ② that if values of b higher than sixty are desired then the number of terms to be summed in such equations as (48) and (64) are correspondingly increased. If b were made excessively large the procedure given here would be inefficient.

For easy reference, the basic formulas used in the program are given again here with the same equation number they carried previously.

$$I_x(a,b) = 1 - I_{1-x}(b,a) = 1 - \bar{I}_x(a,b) , \quad (5)$$

$$I_x(a,b) = \sum_{i=1}^j x^a \frac{\Gamma(a+i-1)}{\Gamma(a) \Gamma(i)} (1-x)^{i-1} , \quad b = j . \quad (48)$$

$$I_x(a,b) = I_x(a,1/2) + \sum_{i=1}^{j-1} x^a \frac{\Gamma(a+i-1/2)}{\Gamma(a) \Gamma(i+1/2)} (1-x)^{i-1/2} ,$$

$$\begin{cases} a = k - \frac{1}{2} \\ b = j - \frac{1}{2} \end{cases} . \quad (59)$$

$$I_x(a,1/2) = I_x(1/2, 1/2) - \sqrt{x} \sqrt{1-x} \sum_{i=1}^{k-1} \frac{\Gamma(i)}{\Gamma(i+1/2) \Gamma(1/2)} x^{i-1} ,$$

$$b = 1/2, a = k - 1/2 \leq 60 . \quad (60a)$$

$$I_x(1/2,b) = I_x(1/2, 1/2) + \sqrt{x} \sqrt{1-x} \sum_{i=1}^{j-1} \frac{\Gamma(i)}{\Gamma(i+1/2) \Gamma(1/2)} (1-x)^{i-1} ,$$

$$a = \frac{1}{2}, b = j - \frac{1}{2} . \quad (60b)$$

$$I_x\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{2}{\pi} \tan^{-1} \left(\frac{\sqrt{x}}{\sqrt{1-x}} \right). \quad (61)$$

$$\lambda(\epsilon'') = \left[1 - \left(\frac{M\sqrt{a-1}}{\sqrt{\pi}} \epsilon'' \right)^{\frac{1}{a-1}} \right]^{1/2} \rightarrow \left[-\frac{1}{a-1} \ln \epsilon'' \right]^{1/2}, \quad (a \rightarrow \infty). \quad (80)$$

$$I_x(a, 1/2) \cong (\lambda - \sqrt{1-x}) \frac{\Gamma(a + 1/2)}{\Gamma(a) \Gamma(1/2)}$$

$$\cdot \sum_{i=1}^m w_i \left\{ 1 - \left[(\lambda - \sqrt{1-x}) \left(\frac{1+y_i}{2} \right) + \sqrt{1-x} \right]^2 \right\}^{a-1} + \frac{\epsilon''}{2},$$

$$\lambda^2 < 4(1-x).$$

(95)

$$\bar{I}_x(a, 1/2) \cong \sqrt{1-x} \frac{\Gamma(a + 1/2)}{\Gamma(a) \Gamma(1/2)}$$

$$\cdot \sum_{i=1}^m w_i \left\{ 1 - \left[\sqrt{1-x} \left(\frac{1+y_i}{2} \right) \right]^2 \right\}^{a-1}, \quad \lambda^2 \geq 4(1-x).$$

(96)

$$\ln \Gamma(s) \cong (s - 1/2) \ln(s - 1) - (s - 1) + (1/2) \ln 2\pi$$

$$+ \frac{1}{12} \frac{1}{s-1} - \frac{1}{360} \frac{1}{(s-1)^3} + \frac{1}{1260} \frac{1}{(s-1)^5} - \dots \quad (101)$$

$$\ln \Gamma(a+c) - \ln \Gamma(a) \cong c - \frac{1}{2} \frac{c}{a} \left[\frac{\ln(1+c/a)}{c/a} - 1 \right]$$

$$- \frac{1}{2} \frac{c}{a} + c \ln(a+c) - \frac{1}{12} \left[\frac{1}{a} - \frac{1}{a+c} \right] \quad (102)$$

$$+ \frac{1}{360} \left[\frac{1}{a^3} - \frac{1}{(a+c)^3} \right] - \frac{1}{1260} \left[\frac{1}{a^5} - \frac{1}{(a+c)^5} \right] + \dots,$$

$$\frac{\ln(1 + \theta)}{\theta} - 1 = -y + \frac{2y^2}{2 + \theta} \left[\frac{1}{3} + \frac{y^2}{5} + \frac{y^4}{7} + \frac{y^6}{9} + \dots \right], \quad (103)$$

$$A_p = \left(\frac{2p+1}{2p+3} \right) y^2 A_{p-1}, \quad p = 1, 2, \dots, \quad (104)$$

where

$$A_p \equiv y^{2p} / (2p + 3), \quad A_0 = 1/3. \quad (105)$$

①

MASTER FLOW CHART $I_x(a, b)$

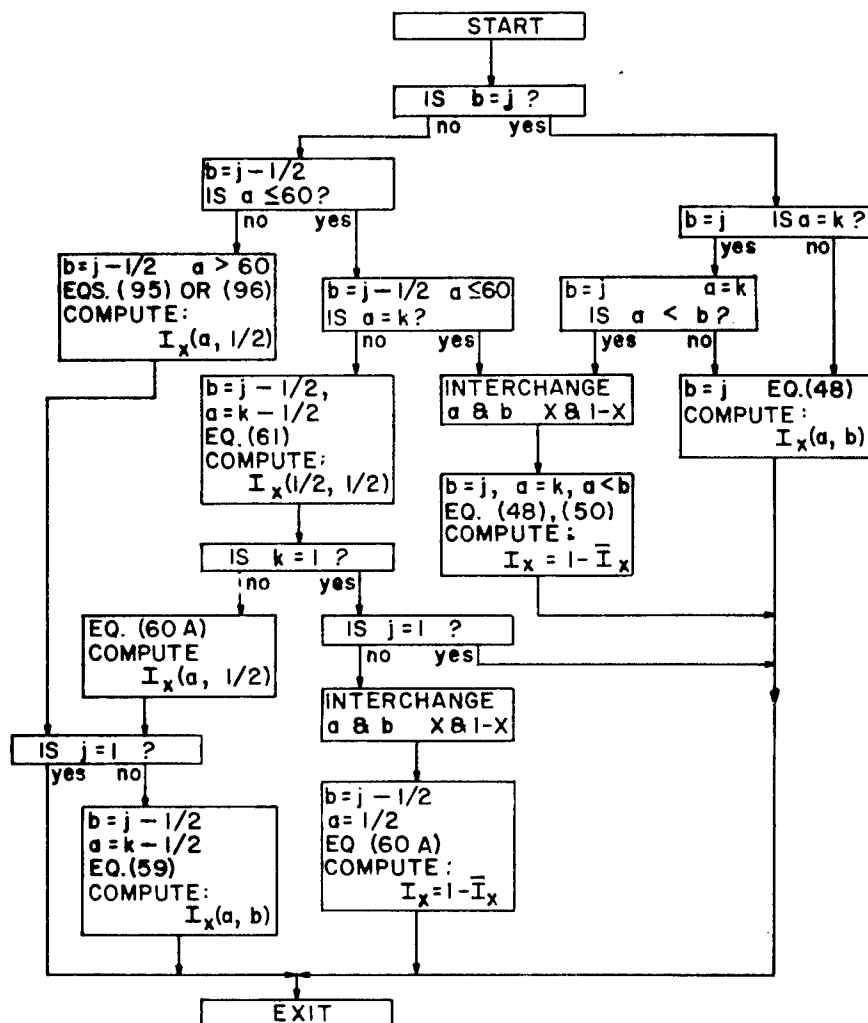
$$I_x(a, b) = 1 - I_{1-x}(b, a) \equiv 1 - \bar{I}_x(a, b)$$

k AND j: POSITIVE INTEGERS

INPUT: a, b, x $\left\{ \begin{array}{lll} 1/2 \leq b \leq 60 & 1/2 \leq a \leq 10^8 & 0 \leq x \leq 1 \\ \epsilon, \epsilon' & b = j \text{ OR } j - 1/2 & a = k \text{ OR } k - 1/2 \end{array} \right.$

OUTPUT: $I_x(a, b)$ $0 \leq I_x(a, b) \leq 1$

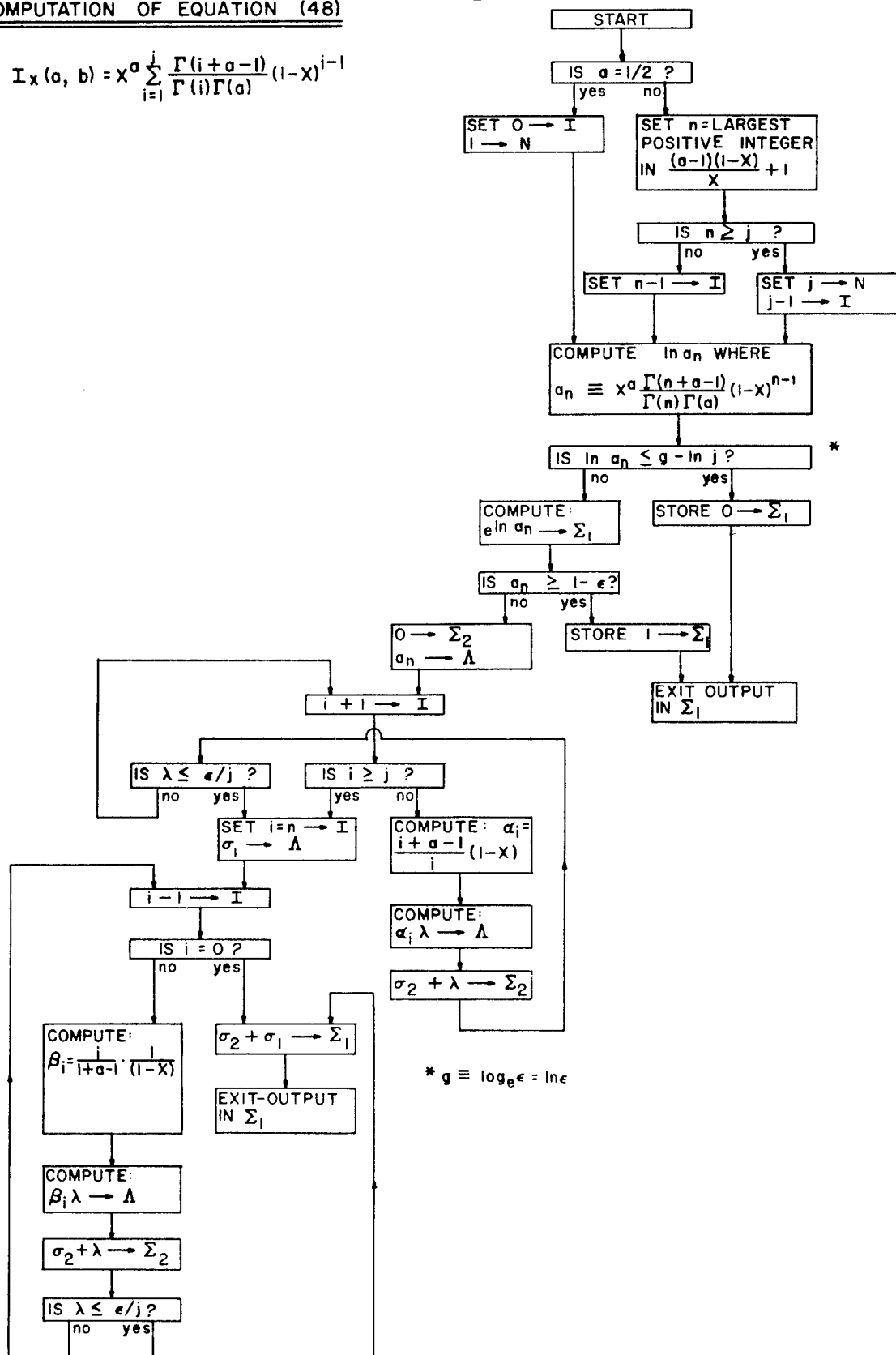
PERTINENT EQUATIONS ARE GIVEN ON PAGES 41 - 43



COMPUTATION OF EQUATION (48)

$$I_x(a, b) = x^a \sum_{i=1}^j \frac{\Gamma(i+a-1)}{\Gamma(i)\Gamma(a)} (1-x)^{i-1}$$

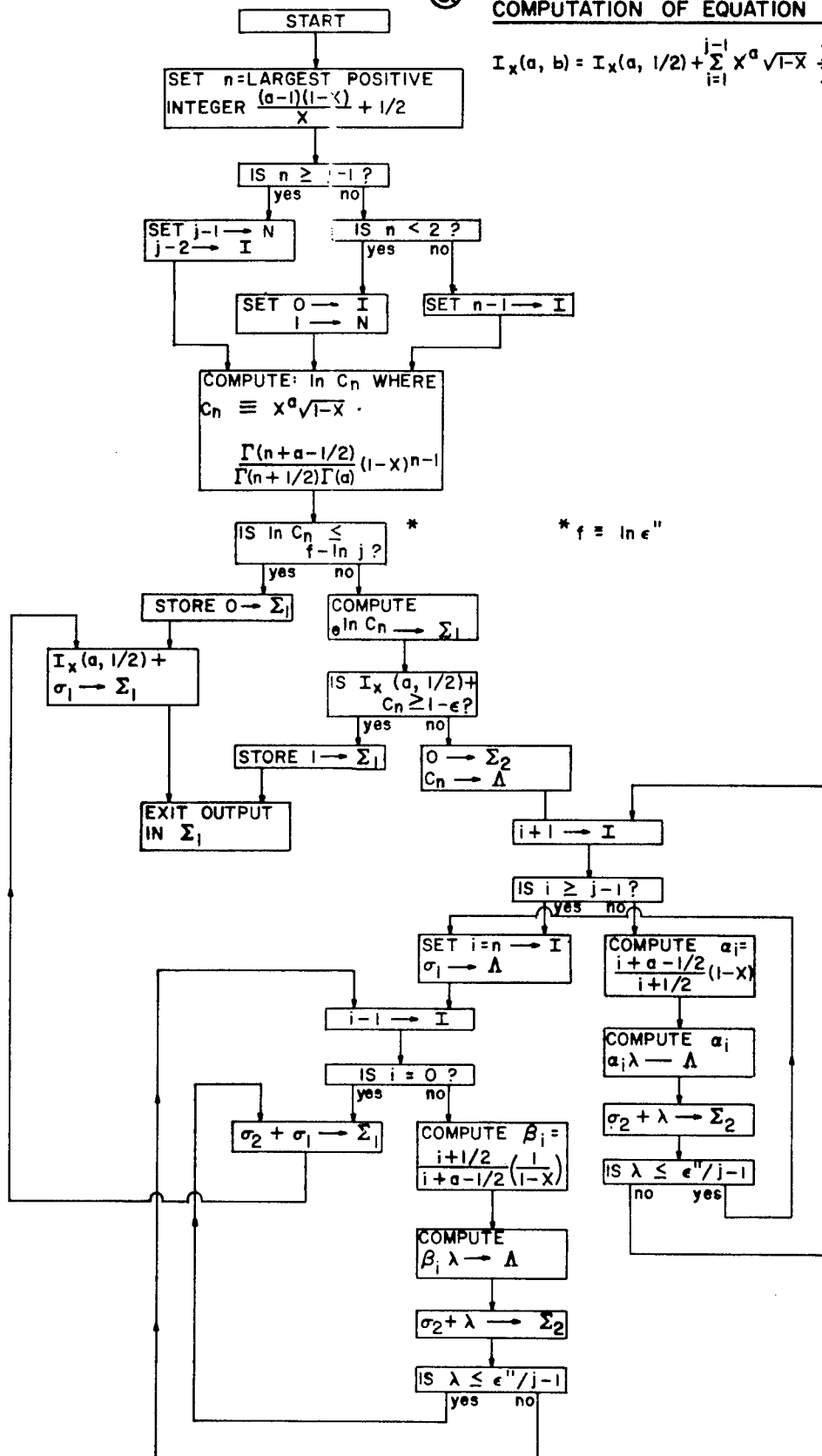
②



③

COMPUTATION OF EQUATION (59)

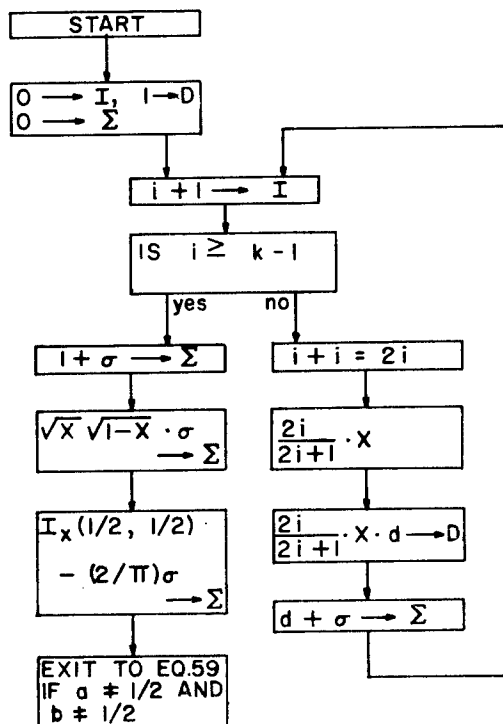
$$I_x(a, b) = I_x(a, 1/2) + \sum_{i=1}^{j-1} x^a \sqrt{1-x} \frac{\Gamma(i+a-1/2)}{\Gamma(i+1/2)\Gamma(a)} (1-x)^{i-1}$$



④

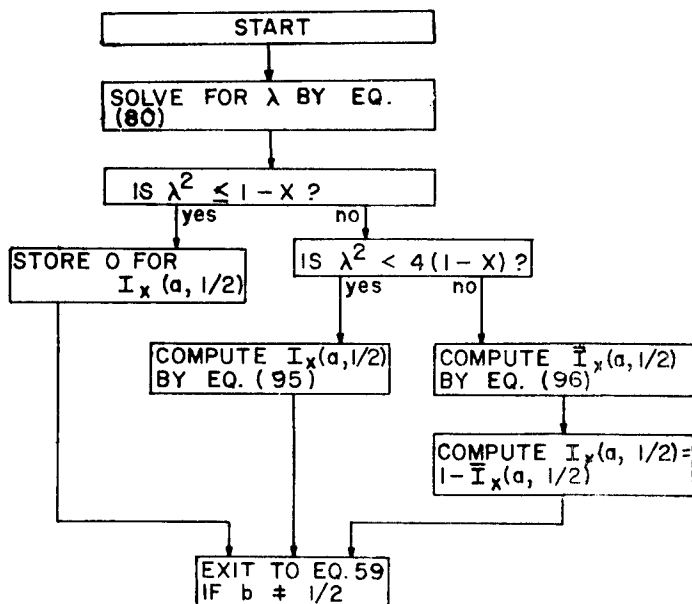
COMPUTATION OF EQUATION 60A

$$I_x(a, 1/2) = I_x(1/2, 1/2) - \sqrt{x} \sqrt{1-x} \sum_{i=1}^{k-1} \frac{\Gamma(i)}{\Gamma(i+1/2)\Gamma(1/2)} x^{i-1}, \quad I_x(1/2, 1/2) = \frac{2}{\pi} \tan^{-1}\left(\frac{\sqrt{x}}{\sqrt{1-x}}\right)$$



⑤

COMPUTATION OF EQUATIONS (95), (96)

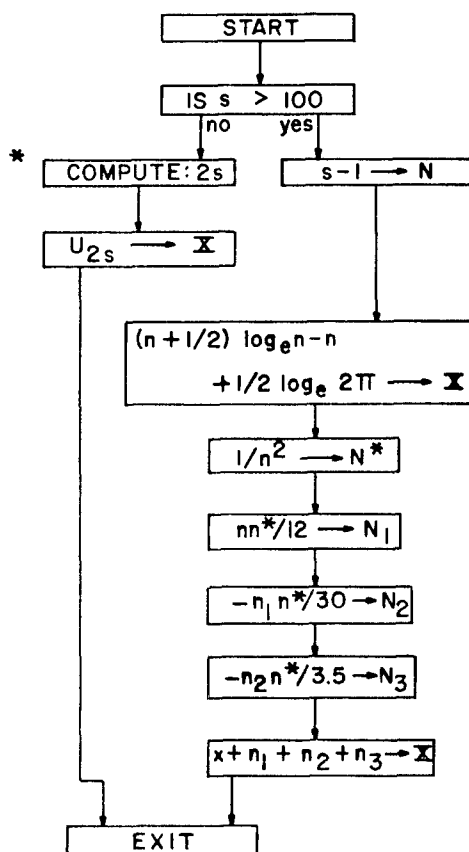


COMPUTATION OF $K \equiv \ln \Gamma(a+c) - \ln \Gamma(a)$
EQUATION 102



⑦

COMPUTATION OF $[\log_e \Gamma(s)]$, EQUATION 101



* VALUES OF $\log_e \Gamma(s)$ STORED IN CONSECUTIVE LOCATIONS U_1, U_2, \dots, U_{200} IN INCREASING ORDER OF s FOR $s=1/2(1/2)100$, e.g., U_{140} CONTAINS $\log \Gamma(70)$.

$1/2 \log_e 2\pi = 0.9189\ 3853\ 3204\ 67$

n_3 IS NOT NEEDED IF $\epsilon \geq 5 \times 10^{-13}$

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APPENDIX A

DERIVATIONS OF:

(18), (19), (20); (48), (57); (59), (60), (61); (69), (70)

In this appendix derivations are given for equations (18), (19), (20); (48), (57); (59), (60), (61); (69), (70).

A. Derivation of (18), (19), (20)

Equation (18) is given by

$$I_x(a, b) = x I_x(a - 1, b) + (1 - x) I_x(a, b - 1) . \quad (18)$$

This equation is proved by first establishing the relation

$$I_x(a + 1, b - 1) = I_x(a, b) - \frac{\Gamma(a + b)}{\Gamma(a + 1) \Gamma(b)} x^a (1 - x)^{b-1} . \quad (107)$$

An integration by parts on $B_x(a, b)$ gives

$$B_x(a, b) = \frac{a - 1}{b} B_x(a - 1, b + 1) - \frac{x^{a-1} (1 - x)^b}{b} . \quad (108)$$

Therefore

$$B_x(a + 1, b - 1) = \frac{a}{b - 1} B_x(a, b) - \frac{x^a (1 - x)^{b-1}}{b - 1} . \quad (109)$$

Multiplying (109) by $\Gamma(a + b)/[\Gamma(a + 1) \Gamma(b - 1)]$ leads directly to (107).

The proof for (18) follows. From (107)

$$\begin{aligned} I_x(a - 1, b) - I_x(a, b - 1) &= \frac{x^{a-1} (1 - x)^{b-1}}{\Gamma(a) \Gamma(b)} \Gamma(a + b - 1) \\ &= \frac{x^{a-1} (1 - x)^{b-1}}{\Gamma(a) \Gamma(b)} \Gamma(a + b) \\ &\quad \cdot \left[1 - \frac{a - 1}{a + b - 1} - \frac{b - 1}{a + b - 1} \right] . \quad (110) \end{aligned}$$

Now assuming $x \neq 0, 1$, (110) may be written as

$$\frac{d}{dx} [I_x(a,b)] = \frac{d}{dx} [x I_x(a-1,b)] + \frac{d}{dx} [(1-x) I_x(a,b-1)] . \quad (111)$$

Carrying out the obvious integration, gives (18) plus an integration constant which can be shown to vanish by letting x tend to zero. It is obvious that (18) also holds for $x = 0$ and $x = 1$, and the proof is complete.

In order to derive (19), the following relation is used:

$$b I_x(a,b+1) + a I_x(a+1,b) = (a+b) I_x(a,b). \quad (112)$$

Equation (112) is proved by writing $B_x(a,b)$ as

$$\begin{aligned} B_x(a,b) &= \int_0^x t^{a-1} (1-t)^{b-1} [(1-t) + t] dt \\ &= B_x(a,b+1) + B_x(a+1,b). \end{aligned} \quad (113)$$

Multiplying (113) by $\Gamma(a+b+1)/[\Gamma(a)\Gamma(b)]$ and using (45) gives (112). The index b is reduced by unity throughout (112); this does not affect the validity of the relation, and subsequently $I_x(a,b)$ as given by (107) is substituted for the second term on the left hand side of (112). The result, after some trivial algebra, is

$$I_x(a,b) = I_x(a,b-1) + \frac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)} x^a (1-x)^{b-1}, \quad b \geq 1, \quad (114)$$

When $b = 1$, $I_x(a,0)$ is to be interpreted as zero.

By applying (5) to (114), or by manipulations similar to those used for deriving (114), another useful result is obtained,

$$I_x(a,b) = I_x(a-1,b) - \frac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^b, \quad a \geq 1, \quad (115)$$

where $I_x(0,b) \equiv 0$ in (115). Equations (19), (20) follow from (107) by setting $b = 3/2$ for (19), and by setting $a = 1/2$ and increasing b by unity throughout (107) for (20). If one subtracts (115) from (114) the result is equivalent to (107).

B. Derivation of (48), (57)

Equation (48) follows easily by writing (114) as a telescoping series, where b is replaced by a running index i , such that

$$\begin{aligned} I_x(a, b) = I_x(a, j) &\equiv \sum_{i=1}^j [I_x(a, i) - I_x(a, i-1)] \\ &= \sum_{i=1}^j x^a \frac{\Gamma(a+i-1)}{\Gamma(a) \Gamma(i)} (1-x)^{i-1}, \end{aligned} \quad (116)$$

which is also (48).

Equation (57) follows easily also by using (45). Thus

$$\begin{aligned} \frac{a_{i+1}}{a_i} &= \frac{x^a (1-x)^i \Gamma(a+i)/[\Gamma(a) \Gamma(i+1)]}{x^a (1-x)^{i-1} \Gamma(a+i-1)/[\Gamma(a) \Gamma(i)]} \\ &= (1-x) \frac{a+i-1}{i}. \end{aligned} \quad (117)$$

C. Derivation of (59), (60), (61)

As in deriving (48), (59) is obtained by writing (114) as a telescoping sum. However in this case, $b = j - 1/2$ and

$$\begin{aligned} I_x(a, b) - I_x(a, 1/2) &\equiv \sum_{i=1}^{j-1} [I_x(a, i+1/2) - I_x(a, i-1/2)] \\ &= \sum_{i=1}^{j-1} x^a \frac{\Gamma(a+i-1/2)}{\Gamma(a) \Gamma(i+1/2)} (1-x)^{i-1/2}, \end{aligned} \quad (118)$$

which is the result desired.

Equation (60b) is directly deducible from (118) by setting $a = 1/2$. The equation (60a) is also easily proven by applying (5) to (60b). The term $I_x(1/2, 1/2)$ is given by

$$I_x(1/2, 1/2) \equiv \{\Gamma(1)/[\Gamma(1/2) \Gamma(1/2)]\} \int_0^x \frac{dt}{\sqrt{t} \sqrt{1-t}}. \quad (119)$$

The transformation

$$t = \sin^2 \theta, \quad 0 \leq \theta \leq \pi/2,$$

applied to the integral of (119) gives for $I_x(1/2, 1/2)$,

$$I_x(1/2, 1/2) = \frac{1}{\pi} 2 \int_0^{\sin^{-1} \sqrt{x}} \frac{\sin \theta \cos \theta}{\sin \theta \cos \theta} d\theta = \frac{2}{\pi} \sin^{-1} \sqrt{x}, \quad (120)$$

which is equivalent to (61).

D. Derivation of (69), (70)

These results are obtained in exactly the same way that (117) was generated.

APPENDIX B

DERIVATIONS OF: (91), (92).
PROOF THAT $U_{n,r}(u) \leq U_{n,r}(0)$, $0 \leq u \leq 1$

In this appendix the three following results are proved:

(A) Equations (91), (92)

(B) The maxima of $|U_{n,r}|$ decrease monotonically as a function of u on the interval $[0,1]$.

A. Proof of Equations (91) and (92)

In (89), $U_{n,r}$ is defined accordingly

$$U_{n,r} = \frac{d^r}{du^r} [(1 - u^2)^n], \quad 0 \leq u \leq 1. \quad (89)$$

Equation (94) states that $U_{n,r}$ is given by

$$U_{n,r} = \sum_{i=0}^{[r/2]} (-1)^{r-i} \frac{2^{r-2i} r! \Gamma(n+1)}{i!(r-2i)! \Gamma(n-r+i+1)} u^{r-2i} (1 - u^2)^{n-r+i}, \quad n \geq r, \quad (91)$$

where $[r/2]$ represents the greatest integer in $r/2$. The proof is by induction. Thus for $r = 0, 1, 2$, (91) is easily seen to be valid. It is necessary to show (91) holds for $U_{n,r+1}$ assuming it holds for $U_{n,r}$; $U_{n,r+1}$ would be given by

$$U_{n,r+1} = \sum_{i=0}^{[r+1/2]} (-1)^{r+1-i} \frac{2^{r-2i+1} (r+1)! \Gamma(n+1)}{i!(r-2i+1)! \Gamma(n-r+i)} \cdot u^{r-2i+1} (1 - u^2)^{n-r+i-1}. \quad (121)$$

The proof follows:

Introduce A_i , such that

$$A_i = (-1)^{r-i} \frac{2^{r-2i} r! \Gamma(n+1)}{i!(r-2i)! \Gamma(n-r+i+1)}, \quad i \geq 0; \quad A_i = 0, \quad i < 0. \quad (122)$$

Now differentiating (91), and subsequently using (122) one obtains

$$\begin{aligned} \frac{d}{du} [U_{n,r}] &= (-1)^{r+1} \frac{2^{r+1} \Gamma(n+1)}{\Gamma(n-r)} u^{r+1} (1-u^2)^{n-r-1} \\ &+ \sum_{i=1}^{[r/2]} [(r-2i+2) A_{i-1} - 2(n-r+i) A_i] u^{r-2i+1} \\ &\cdot (1-u^2)^{n-r+1-1} + (r-2[r/2]) A_{[r/2]} u^{r-2[r/2]} \\ &\cdot (1-u^2)^{n-(r-[r/2])}. \end{aligned} \quad (123)$$

The constant factor under the summation sign of (123) can be simplified. Thus

$$(r-2i+2) A_{i-1} - 2(n-r+i) A_i$$

is equal to

$$\begin{aligned} &(-1)^{r-i+1} \frac{2^{r-2i+2} r! \Gamma(n+1)}{(i-1)!(r-2i+1)! \Gamma(n-r+i)} \\ &+ (-1)^{r-i+1} \frac{2^{r-2i+1} r! \Gamma(n+1)}{i!(r-2i)! \Gamma(n-r+i)} \\ &= (-1)^{r-i+1} \frac{2^{r-2i+1} (r+1)! \Gamma(n+1)}{i!(r-2i+1)! \Gamma(n-r+i)}. \end{aligned} \quad (124)$$

The last term in (123) is equal to the $\left[\frac{r+1}{2}\right]$ term of (121) for r odd. The $\left[\frac{r+1}{2}\right]$ term of (121) for r even is included in the $[r/2]$ term of the sum in (123). It therefore follows that (123) is equal to (121), and the proof is complete.

Equation (92) can be derived as follows:

$$U_{n+1,r+2} = \frac{d^{r+2}}{du^{r+2}} [(1 - u^2)^{n+1}] = \frac{d^{r+2}}{du^{r+2}} [(1 - u^2) (1 - u^2)^n], \quad (125)$$

and by Leibnitz's rule for obtaining the $(r + 2)^{th}$ derivative of a product, it is easily shown that

$$U_{n+1,r+2} = (1 - u^2) U''_{n,r} - 2(r + 2) u U'_{n,r} - (r + 1)(r + 2) U_{n,r}. \quad (126)$$

However

$$\begin{aligned} U_{n+1,r+2} &= \frac{d^{r+1}}{du^{r+1}} \left[\frac{d}{du} (1 - u^2)^{n+1} \right] = \frac{d^{r+1}}{du^{r+1}} [-2(n + 1) u(1 - u^2)^n] \\ &= -2(n + 1) \frac{d^{r+1}}{du^{r+1}} [u(1 - u^2)^n] \\ &= -2(n + 1) [u U'_{n,r} + (r + 1) U_{n,r}], \end{aligned} \quad (127)$$

where Leibnitz's rule was employed again to obtain the last equation. Subtracting both sides of (127) from both sides of (126) gives

$$(1 - u^2) U''_{n,r} + 2(n - r - 1) u U'_{n,r} + (r + 1)(2n - r) U_{n,r} = 0, \quad (92)$$

which is equation (92).

B. The absolute values of the extrema of (89) decrease as a function of u on $[0, 1]$ provided $n > r + 1/2$. The proof for this statement was suggested by techniques used by Szego in [13, Chapter VII].

Consider a function f such that

$$f \equiv A y^2(u) + \phi(u)(y')^2, \quad (128)$$

where

A is a positive constant, and $\varphi(u)$ is non-negative for u in $[0,1]$. Therefore

$$f \geq 0,$$

and

$$f' = y' [2\varphi y'' + \varphi' y' + 2Ay] . \quad (129)$$

Now let

$$y \equiv U_{n,r} , \quad \varphi(u) = \frac{1}{2} (1 - u^2) , \quad 2A \equiv (r + 1) (2n - r) . \quad (130)$$

Then, after substituting (92) into (129)

$$f' = 2y' [- (n - r - 1/2) uy'] , \quad 0 \leq u \leq 1 . \quad (131)$$

Therefore it is concluded

$$f' < 0 \quad \text{if} \quad n > r + 1/2, u \neq 0, y' \neq 0 , \quad (132)$$

where the inequality of (132) also insures A as defined in (130) to be positive. Since f' is negative on $(0,1)$ at points where $y' \neq 0$, this means that f is a decreasing function of u on $[0,1]$. The clinching argument follows by considering those values of u for which $y'(u) = 0$, on $[0,1]$, i.e., the extrema points of y , which we call u_m . For such u , (128) can be written as

$$y^2(u_m) = f(u_m)/A . \quad (133)$$

But since f is a decreasing function of u , then $y^2(u_m)$ cannot increase as the u_m increase from 0 to 1. If r is even, $u = 0$ is a point of the set $\{u_m\}$, because it is evident from (91) that $y'(0) = U_{n,r+1}(0) = 0$. If r is odd, $u = 0$ does not belong to the set $\{u_m\}$. Thus

$$|U_{n,r}(0)| > |U_{n,r}(u)|, \quad 0 < u \leq 1, r \text{ even} . \quad (134)$$

The result which has just been proved can also be deduced directly from a theorem given by Tricomi, [16; p 99]. The theorem essentially states that if a differential equation has the form

$$\frac{d}{du} \left[p(u) \frac{dy}{du} \right] + P(u) y = 0, \quad (135)$$

such that

- a) $p(u)$ and $P(u)$ and their first derivatives are continuous on (a, b) , i.e., $p(u)$, $P(u)$ are in C' on (a, b) ,
- b) $[p(u)P(u)]$ is a non-decreasing (non-increasing) function of u in (a, b) ,
- c) $P(u) \neq 0$ in (a, b) ,

then the maxima and minima which occur in (a, b) of any integral $y(u)$ of (135) are such that the corresponding values of $|y|$ form a non-increasing (non-decreasing) sequence. If the hypotheses of this theorem are satisfied on the half-open interval $[a, b)$, then it is easily shown by going through Tricomi's proof step by step that the conclusion holds on the half-open interval, that is, the extrema on $[a, b)$ are such that the corresponding values of $|y|$ form a non-increasing (non-decreasing) sequence.

Equation (92) is easily put in the form of (135), (see [16; p 96]), so that (92) becomes

$$\frac{d}{du} \left[(1 - u^2)^{-(n-r-1)} \frac{dy}{du} \right] + (r+1)(2n-r)(1 - u^2)^{-(n-r)} y = 0, \quad (136)$$

where

$$p(u) \equiv (1 - u^2)^{-(n-r-1)}, \quad P(u) = (r+1)(2n-r)(1 - u^2)^{-(n-r)}. \quad (137)$$

On $[0, 1)$, $p(u)$ and $P(u)$ are obviously in C' , $P(u) \neq 0$, and $[p(u)P(u)]$ is non-decreasing provided $n \geq r + 1/2$. Therefore the hypotheses of the modified theorem (for the half-open interval) are satisfied, and the conclusion of the modified theorem holds and implies the result which was to be proved.

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<p>An efficient method is given for computing the incomplete beta function ratio, $I_x(a,b)$, on a high speed digital computer. The arguments a, b, are limited to positive integral multiples of one-half values over the ranges $1/2 \leq a \leq 10^8$, $1/2 \leq b \leq 60$.</p> <p>The program has been coded in STRAP for the IBM 7030 (STRETCH). The average computing time for a ten decimal digit value of $I_x(a,b)$ is 2.6 milliseconds; on an IBM 7090 it would be about 8 milliseconds per case.</p>			

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